A Study of Game Theory: on Solutions and Applications in Cooperative Games

— Nash Bargaining Solution, the Bargaining Set Family and Other Cooperative Solutions

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Abstract

Ever since the ground-breaking work *Theory of Games and Economic Behavior* by Von Neumann and Morgenstern more than sixty years ago, game theory has become an increasingly important tool for abstracting and analyzing real-world problems involving conflict and cooperation among different parties. There are two major types of games, zero-sum and non-zero-sum games; the latter is then further divided into non-cooperative and cooperative games. This paper focuses on solutions for cooperative games. In particular, I studied in detail the Nash bargaining solution and the bargaining set family. After proofs of some existence and uniqueness theorems, I compared and contrasted the assumptions and rationale behind different solution concepts, and evaluated their applicability using some examples and a case study modeling the airliner market.

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Chapter 1

Introduction and Background

Game theory is a relatively new branch of mathematics that deals with the analysis of "games" — not so much in the sense of poker or chess games, but more in terms of situations that involve conflict and cooperation among the players. In essence, game theory is a mathematical method for analyzing strategic interactions. When oligopolists in a market make product price and quantity decisions, when institutions negotiate on trading contracts and partnership terms, when countries make threats to go to war against each other, each agent has to be concerned about the reactions and expectations of the other parties involved. The incorporation of these reactions (to the best of one's knowledge) into the strategic decision making process calls for the study and application of game theory.

The foundation of game theory was set up in a collaborative study by John von Neumann and Oskar Morgenstern in their ground-breaking book, *Theory* *of Games and Economic Behavior* in 1944 [36]. The book was a landmark in the transition of the study of games in economics (such as the study of oligopoly equilibrium by Auguste Cournot) into a mathematical discipline, with more precision and a stronger sense of objectivity. The most important idea introduced by von Neumann and Morgenstern was their detailed analysis of two-person zero-sum games. They showed that in zero-sum games, games in which the goals of the players are always strictly opposed to one another, there always exists a mixed-strategy minimax solution.

In the following decade, game theory studies flourished at Princeton University and the RAND Corporation under the sponsorship of the Air Force. John Nash formulated a universal solution concept for non-cooperative games by introducing and proving the existence of what is now known as the "Nash Equilibrium." Based on Nash's work, Reinhard Selten further reduced the number of possible Nash equilibria by proposing the concept of "subgame perfect equilibrium." The story of "the Prisoner's Dilemma," invented by A. W. Tucker, became part of the popular culture as a vivid example of the interplay between competition and cooperation. It took academia about two decades to overcome the stereotypical view that game theory was merely a study of two-person zero-sum games whose application was restricted to military situations. By 1994, when the Nobel Prize in Economics was awarded to Nash, Selten and Harsanyi (whose major contribution was the study of games with incomplete information), game theory had finally obtained a central position in economics — in particular, non-cooperative game theory, studies on games that exclude pre-play binding agreements, had made a great impact on economics research, especially in the realm of industrial organization.

In the meantime, cooperative game theory has developed from a few rough ideas into an important chapter of game theory with deep mathematical theorems. Allowing pre-play communication and binding agreements, the study of cooperative games enhances our understanding of the conditions for successful cooperation — What makes some coalitions stable and others vulnerable? What is fairness when dividing revenues among players of different power positions? How can we reduce the feasible outcomes to complicated cooperative games? Many mathematicians and scholars in other disciplines have come up with different solution concepts based on varied assumptions, in an effort to better model the real world problems. Cooperative game theory has been widely used to analyze warfare, political choices of presidential candidates, allocation problems of social resources, as well as many other areas in the social sciences.

My paper will be structured in a similar way as the evolution of game theory. In Chapter 2, I will give a very brief introduction to Utility Theory, a theoretical support of game theory. In Chapter 3, zero-sum games are studied and the Minimax Theorem is proved. Chapter 4 is devoted to non-cooperative games: I will give a brief account of the Nash Equilibrium, and provide examples to illustrate the different types of two-person non-cooperative games.

The main focus of my paper is on cooperative games (Chapter 5). As it is still a developing area of research with an abundance of solution concepts, I am interested in studying and comparing these various intricate solutions, and see under what circumstances some solutions might be preferred to others. I will give a detailed account of two major solution concepts introduced by two Nobel Laureates, the *Nash bargaining solution* by John Nash, and the *bargaining set* by Robert Aumann[1]. The latter concept gradually evolved into what I call the *bargaining set family*, which include the *kernel*, the *nucleolus* and modifications of these concepts. The major theorems for these solutions are provided, among which are some proofs of lemmas, and theorems for which alternative proofs I derived . After a brief examination of a few other important solution concepts for cooperative games, I will compare and evaluate these different solutions based on their rationale, assumptions and respective range of applicability.

In Chapter 6, a brief literature review is provided, after which I will construct a simplified model of the airliner market as a three-person cooperative game. This case study further illustrates the interrelationship and differences among the several solution concepts studied, and it also gives rise to some non-trivial implications. Chapter 7 concludes the paper, and contains recommendations of avenues for further research.

The theorems, lemmas and examples in the paper are numbered in the form of (chapter.section.sequence number). Since a lot of new definitions are introduced here, for the reference convenience of the reader, I have included an index at the very end of the paper.

[1]Robert Aumann won the Nobel Prize in Economics in 2005, along with Thomas Schelling, for "having enhanced our understanding of conflict and cooperation through game-theory analysis."

Chapter 2

Utility Theory

In most, if not all, books about game theory, utility theory is introduced and described as a theoretical support. Interestingly, game theory was developed before utility theory: the latter was indeed created as a pillar for the former. Even though utility theory can stand on its own and is useful in other fields, the close inter-relationship between these two areas warrants a brief introduction of utility theory, before we delve fully into the world of game analysis.

Utility theory assigns numbers (or functions) to various alternatives that a person is facing, according to how much "utility", or benefit, each alternative brings. Like game theory, it tries to abstract mathematical relationships and characteristics from concrete real-world choices.

Let us first introduce some notation for utility theory.

2.1 Notation

Suppose a person is facing r basic alternatives, A_1, A_2, $\cdots$ A_r, from the compositions of which he makes a choice. We define a *single lottery* L that yields each alternative A_i with p_i as

$$L = (p_1 A_1, p_2 A_2, \cdots, p_r A_r),$$

where p_i is the objective probability assigned to the occurrence of alternative A_i:

$$\sum_{i=1}^{r} p_i = 1 \quad \text{and} \quad p_i \geq 0 \quad \text{for } i = 1, 2 \cdots r$$

A *compound lottery* L' is constructed based on a finite number of single lotteries:

$$L' = (q_1 L_1, q_2 L_2, \cdots, q_r L_r),$$

where q_i is the probability associated with lottery L_i.

To make a decision on the most preferred lottery, we first have to define the notion of "preference" and "indifference":

Definition Assume a person has an individual taste system that is consistent overtime. If given two alternatives A_i and A_j, he always chooses A_i over A_j, then A_i is *preferred to* A_j, denoted as $A_i \succ A_j$. If he equally likes the two alternatives, then A_i is *indifferent to* A_j, so $A_i \sim A_j$. If A_j is not preferred to A_i, then $A_i \succeq A_j$.

The preference and indifference relationship between two lotteries L_1 and L_2 can be defined similarly.

Now we introduce the Utility Theorem and some of its implications in game theory.

2.2 Utility Theorem

Given r basic alternatives A_1, A_2, $\cdots$, A_r, assume the following (see Luce and Raiffa [11, Chapter 2] for a detailed explanation):

1. Ordering of alternatives:

   The preference and indifference relationship holds between any two basic alternatives, and is transitive.

2. Reduction of compound lotteries:

   Each compound lottery is indifferent to a single lottery that is composed of basic alternatives.

3. Continuity:

   Each basic alternative is indifferent to a lottery involving a least preferred alternative and a most preferred alternative.

4. Substitutability:

   A lottery indifferent to a basic alternative A_i can be substituted with A_i in any lottery.

5. Transitivity among lotteries:

   The preference and indifference relationships among lotteries are also transitive.

6. Monotonicity:

   If $A_1 \succeq A_r$, then a lottery $(pA_1, (1-p)A_r)$ is preferred or indifferent to $(p'A_1, (1-p')A_r)$, if and only if $p \geq p'$.

With these six axioms above, we have the following:

Theorem 2.2.1 Utility Theorem *Given two lotteries L and L', let $L = (p_1A_1, \cdots, p_rA_r)$ and $L' = (p_1'A_1, \cdots, p_r'A_r)$. There are real numbers u_i associated with the basic alternatives A_i $(i = 1, 2, \cdots, r)$, such that for L and L', the magnitudes of the value of the functions*

$$u(L) = p_1u_1 + p_2u_2 + \cdots + p_ru_r \quad \textit{and} \quad u(L') = p_1'u_1 + p_2'u_2 + \cdots + p_r'u_r$$

reflect the preference between L and L'.

The function $u(L)$ is defined as a *utility function*:

Definition Given a set of lotteries $\mathbb{L}$, if the preference and indifference relationship "$\succeq$" satisfies the six axioms, then the *utility function* u: $\mathbb{L} \to \mathbb{R}$ is such that for any two lotteries L and L' in $\mathbb{L}$,

$$u(L) \geq u(L'), \quad \text{if and only if} \quad L \succeq L'.$$

Suppose $u(L)$ is a utility function, and $u'(L) = au(L) + b$ with $a, b \in \mathbb{R}$ and $a > 0$, then

$$L \succ L' \Leftrightarrow u(L) > u(L') \Leftrightarrow au(L) + b > au(L') + b \Leftrightarrow u'(L) > u'(L')$$

Thus, it is shown that if $u(L)$ is a utility function, then a positive linear transformation of $u(L)$ is also a utility function preserving the preference sequence of the lotteries. It can also be shown (see [36, Chapter 3]) that if both $u'(L)$ and $u(L)$ are linear utility functions representing the ordering of $\succeq$, there exist $a, b \in \mathbb{R}$ and $a > 0$, such that

$$u'(L) = au(L) + b$$

Therefore, the uniqueness of a utility function is defined up to linear transformations with a positive coefficient.

This gives rise to problems in interpersonal comparison of utility, which has always been a target for criticism in the application of game theory analysis to group decision making. For instance, in order to select an option that maximizes the group welfare, we inevitably have to "sum up," albeit not necessarily in the traditional sense of addition, the utilities for each individual. Here the assumption is that there is a way to compare interpersonal utilities — there exists some kind of common measuring stick that we can use to assess the utility of an alternative to different people. The most widely used measurement unit, money, seems to be an obvious choice. However, does one dollar mean the same thing to a rich person as to a poor person? Although both would prefer a ten-dollar bill to a one-dollar bill, (thus both of their utility functions yield a higher value when the alternative is ten dollars), most of us would reckon that for the rich, the utility difference between the two options may be negligible; while for the poor, the difference could be considerable. Therefore, since the uniqueness of a utility function is only defined up to a linear transformation, even with money as a unit, the addition of utilities between different people warrants more justification. With more complicated options that cannot be measured in monetary terms, interpersonal comparison is even harder to justify.

With the complexity explained, in most parts of this paper, we assume that interpersonal comparison of utility is possible. We will also introduce some solution concepts that do not require such comparison.

Chapter 3

Zero-Sum Games

There are two major types of games, *zero-sum games* and *non-zero-sum games*. The mechanism and strategies behind these two types of games are quite different: zero-sum games, by definition, require that the payoffs to all the players involved add up to zero. We assume in this paper that each player is rational, i.e. he attempts to maximize his individual payoff. Even though in an n-person game, people may align their welfare and form small coalitions, they will still have to face the question of dividing a specific payoff within that coalition. Therefore, the goals of the players in a zero-sum game are strictly opposed to one another. Each person tries to maximize his own payoff, and in doing so, he also minimizes that of the other players, since the total amount is fixed. In this sense, zero-sum games are competitive in nature.

By contrast, non-zero-sum games are more cooperative than competitive. Since different options give varying total payoffs, it leaves more room for players to achieve mutually beneficial outcomes. Within non-zero-sum games, there exists two subcategories: *cooperative games* and *non-cooperative games*. In non-cooperative games, total payoffs vary for different outcomes but pre-play communication is still forbidden; whereas in cooperative games, players are allowed to have pre-play discussion and make binding agreements. We recognize that more often than not, many real world problems can be better reflected and modeled by non-zero-sum cooperative games; at the same time, the cooperative nature of such games makes it harder to give one single justified and applicable solution, as in the case of non-cooperative games. In this section, we start with the easiest of the three, zero-sum games.

3.1 Remarks on Zero-Sum

Notice that while strictly competitive games are called "zero-sum" games, they indeed constitute all games whose payoffs to all players add up to a constant value. We give a brief note here about why constant-sum games are equivalent to zero-sum games, by introducing the notion of "sunk cost."

In economics terms, sunk costs are costs that cannot be retrieved. For instance, the cost of building a plant cannot be retrieved, even when the owner stops production. Therefore, such costs are foregone, and should not be taken into consideration while making strategic decisions. For a constant-sum game with the payoffs adding up to the amount of m, we can invent a pre-game in which all n players are required to pay a fixed participation fee m/n. Since the fee is a sunk cost, it doesn't affect the strategies that the players choose in the game; meanwhile, we have transformed the game into a zero-sum game, in which all final net payoffs now add up to 0. In this chapter, we proceed with games in which payoffs sum to zero, but bear in mind that the results derived are applicable to all constant-sum games.

We will focus on two-person zero-sum games, and the results can be readily generalized into n-person games.

3.2 Two-Person Zero-Sum Games and the Minimax Theorem

Before we propose the Minimax Theorem, which is fundamentally the most important theorem in the study of zero-sum games, we will first introduce some terminology needed for understanding of the games.

3.2.1 Notation

There are two people playing a game, whom we call the *players*. Both players are faced with a finite number of options, called the *strategies*, from which they need to choose simultaneously. Since both players make their choices at the same time, the result, called the *outcome*, is determined by the combined choice of the players.

There are two types of strategies, *pure strategies* and *mixed strategies*. Pure strategies are like the "basic alternatives" in utility theory, whereas mixed strategies are like "lotteries," i.e. probabilistic combinations of the pure strategies. More specifically, we denote the m pure strategies that player 1 faces by $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$, and the n pure strategies that player 2 faces by $\{\beta_1, \beta_2, \cdots, \beta_n\}$.

Let $a_{ij} = M(\alpha_i, \beta_j) \in \mathbb{R}$ denote the payoff that player 1 gets from player 2. Since it is a zero-sum game, the payoff that player 2 gets is naturally $-a_{ij}$. Thus an outcome $(a_{ij}, -a_{ij})$ is associated with the pair of pure strategies (α_i, β_j).

We denote a mixed strategy employed by player 1 by $\mathbf{x} = (x_1\alpha_1, x_2\alpha_2, \cdots, x_m\alpha_m)$, where

$$\sum_{i=1}^{m} x_i = 1 \quad \text{and} \quad x_i \geq 0 \quad \text{for} \quad i = 1, 2, \cdots, m,$$

which means pure strategy α_i is employed with probability x_i. Similarly, we denote by $\mathbf{y} = (y_1\beta_1, y_2\beta_2, \cdots, y_n\beta_n)$ a mixed strategy employed by player 2, where

$$\sum_{j=1}^{n} y_j = 1 \quad \text{and} \quad y_j \geq 0 \quad \text{for} \quad j = 1, 2, \cdots, n.$$

The payoff for a pair of mixed strategies is defined as

$$M(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j a_{ij}.$$

Notice that by setting $x_j = 0$ for all $j \neq i$, $\mathbf{x}$ can also represent the pure strategy x_i, and similarly for $\mathbf{y}$. Thus, player 1 chooses a strategy $\mathbf{x}$ (pure or mixed) in order to maximize his payoff $M(\mathbf{x}, \mathbf{y})$; simultaneously, player 2 chooses a strategy $\mathbf{y}$ to minimize $M(\mathbf{x}, \mathbf{y})$.

Since the game is purely competitive, it is reasonable to assume that each player is antagonistic, so one needs to be prepared for the worst-case scenario. The goal of maximizing one's payoff therefore translates into getting the biggest return possible in these worst-cast scenarios, which leads to the definition of *security level*.

Definition The *security level* of a player is the least amount that he can receive from a strategy choice, regardless of his opponent's strategy. Let v_1 be the security level for player 1 with strategy $\mathbf{x}$, thus

$$v_1(\mathbf{x}) = \min_{y} M(\mathbf{x}, \mathbf{y})$$

Definition The *maximin strategy* for player 1 is a strategy $\mathbf{x^0}$ that maximizes the security level of player 1:

$$v_1(\mathbf{x^0}) = \max_{\mathbf{x}} v_1(\mathbf{x}) = \max_{\mathbf{x}} \min_{\mathbf{y}} M(\mathbf{x}, \mathbf{y})$$

Let $v_I = v_1(\mathbf{x^0})$. We call v_I the *maximin value* of player 1 in the game.

Since the game is zero-sum, we interpret player 2's aim as the minimization of player 1's payoff, rather than the maximization of his own payoff. Thus, the largest payoff that player 1 can possibly get, when player 2 chooses to play strategy $\mathbf{y}$, is

$$v_2(\mathbf{y}) = \max_{\mathbf{x}} M(\mathbf{x}, \mathbf{y})$$

Analogous to the maximin strategy of player 1, we define the following:

Definition The *minimax strategy* for player 2 is a strategy $\mathbf{y^0}$ that minimizes $v_2(\mathbf{y})$:

$$v_2(\mathbf{y^0}) = \min_{\mathbf{y}} v_2(\mathbf{y}) = \min_{\mathbf{y}} \max_{\mathbf{x}} M(\mathbf{x}, \mathbf{y})$$

Let $v_{II} = v_2(\mathbf{y^0})$. We call v_{II} the *minimax value* of player 2 in the game.

By playing $\mathbf{y^0}$, player 2 guarantees that player 1 will not get more than v_{II}; in other words, player 2 has maximized his security level at $-v_{II}$.

3.2.2 The Minimax Theorem

The Minimax Theorem tells the relationship between the two security levels, v_I and v_{II}, of the two players.

Theorem 3.2.1 Minimax Theorem *In a two-person zero-sum game, the maximin value of player* 1 *is the same as the minimax value of player* 2*; that is*

$$v_I = \max_{\mathbf{x}} \min_{\mathbf{y}} M(\mathbf{x}, \mathbf{y}) = \min_{y} \max_{x} M(\mathbf{x}, \mathbf{y}) = v_{II}$$

We call $v = v_I = v_{II}$ the **value** *of the game.*

The Minimax Theorem looks surprisingly trivial: the commutativity of the operators *max* and *min* seems almost intuitive for any function. Therefore, before delving into a rigorous proof of the Minimax Theorem, we take a brief diversion and first look at a counter example to show that for a non-continuous function $f(x, y)$, the commutativity doesn't always hold.

Consider the following function:

$$f(x,y) \quad = \quad \begin{cases} 2 & x \in \mathbb{Q},\ y \in \mathbb{Q} \\ 4 & x \notin \mathbb{Q},\ y \in \mathbb{Q} \\ 3 & x \in \mathbb{Q},\ y \notin \mathbb{Q} \\ 1 & x \notin \mathbb{Q},\ y \notin \mathbb{Q}, \end{cases}$$

where $x, y \in [0,2]$ and $\mathbb{Q}$ denotes the set of rational numbers.

By some simple algebra, we get

$$2 = \max_x \min_y f(x,y) \neq \min_y \max_x f(x,y) = 3$$

We defined the function $f(x,y)$ to be the notorious function that is not continuous anywhere in the domain. In fact, if we replace $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ with any two disjoint real number intervals, the commutativity of *max* and *min* doesn't hold. We can also replace the payoff $1, 2, 3, 4$ with other numbers or functions, as long as they preserve the order of magnitudes in our example.

3.3 Proof of the Minimax Theorem

Historically, there have been several proofs of the Minimax Theorem, including Von Neumann and Morgenstern (see [36, page 153]), Nash [20], and Owen (see [23, Chapter 2]). We will describe Owen's proof in detail here.

Lemma 3.3.1 $v_I = \max_{\mathbf{x}} \min_{\mathbf{y}} M(\mathbf{x},\mathbf{y}) \leq \min_{\mathbf{y}} \max_{\mathbf{x}} M(\mathbf{x},\mathbf{y}) = v_{II}$

Proof Let $\mathbf{x^0}$ be a maximin strategy of player 1. Let $\mathbf{y^0}$ be a minimax strategy of player 2. Thus

$$v_I = \max_{\mathbf{x}} \min_{\mathbf{y}} M(\mathbf{x},\mathbf{y}) \leq \max_{\mathbf{x}} M(\mathbf{x},\mathbf{y^0}) = M(\mathbf{x^0},\mathbf{y^0})$$

At the same time,

$$v_{II} = \min_{\mathbf{y}} \max_{\mathbf{x}} M(\mathbf{x}, \mathbf{y}) \geq \min_{\mathbf{y}} M(\mathbf{x^0}, \mathbf{y}) = M(\mathbf{x^0}, \mathbf{y^0})$$

Thus,

$$v_I \leq M(\mathbf{x^0}, \mathbf{y^0}) \leq v_{II}.$$

■

Lemma 3.3.2 *If* $\mathbf{x} = (x_1, x_2, \cdots, x_n) \notin B$*, where* B *is a compact and convex set of points in an* n*-dimensional Euclidean space, then there exist real numbers* $p_1, p_2, \cdots, p_n, p_{n+1}$*, such that*

$$\sum_{i=1}^{n} p_i x_i = p_{n+1} \quad \textit{and} \tag{3.1}$$

$$\sum_{i=1}^{n} p_i y_i > p_{n+1} \quad \textit{where} \quad \mathbf{y} = (y_1, y_2, \cdots, y_n) \in B \tag{3.2}$$

Proof We know $\mathbf{x} \notin B$. Since B is a compact set, and any continuous real function defined on a compact set has a minimum, there exists a point $\mathbf{z} \in B$ such that $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}|$, for all $\mathbf{y} \in B$. Thus, $\mathbf{z}$ is the point in B that is closest to $\mathbf{x}$.

Let $p_i = z_i - x_i$ for $i = 1, 2, \cdots, n$. To satisfy the first condition in the lemma, let

$$p_{n+1} = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} (z_i - x_i) x_i = \sum_{i=1}^{n} z_i x_i - \sum_{i=1}^{n} x_i^2$$

Now we have

$$\begin{aligned}\sum_{i=1}^{n} p_i z_i - p_{n+1} &= (\sum_{i=1}^{n} z_i^2 - \sum_{i=1}^{n} z_i x_i) - (\sum_{i=1}^{n} z_i x_i - \sum_{i=1}^{n} x_i^2) \\ &= \sum_{i=1}^{n} (z_i - x_i)^2 > 0 \quad (\mathbf{z} \neq \mathbf{x})\end{aligned}$$

Thus, we have shown that equation (3.2) holds for $\mathbf{z}$. We need to show that it is also true for all $\mathbf{y} \in B$, which we prove by contradiction.

Suppose for some $\mathbf{y} \in B$, $\sum_{i=1}^{n} p_i y_i \leq p_{n+1}$. Since B is a convex set, we know for $0 \leq r \leq 1$, $\mathbf{w_r} = r\mathbf{y} + (1-r)\mathbf{z} \in B$. Let $f(\mathbf{x}, r)$ denote the square of the distance between $\mathbf{x}$ and $\mathbf{w_r}$:

$$f(\mathbf{x}, r) = d^2(\mathbf{x}, \mathbf{w_r}) = \sum_{i=1}^{n} (x_i - r y_i - (1-r) z_i)^2$$

We take the derivative of $f(\mathbf{x}, r)$ with respect to r, and evaluate it at $r = 0$. With some simple algebra, we get

$$\frac{\partial f}{\partial r}|_{r=0} = 2(\sum p_i y_i - \sum p_i z_i)$$

We have shown that $\sum_{i=1}^{n} p_i z_i > p_{n+1}$, and by assumption $\sum_{i=1}^{n} p_i y_i \leq p_{n+1}$, thus, $\frac{\partial f}{\partial r}|_{r=0} < 0$.

Since $f(\mathbf{x}, r)$ is a continuously differentiable function of r and is strictly decreasing in the neighborhood of 0, we know that there exists $\sigma > 0$ such that if $0 < r < \sigma$, then

$$d^2(\mathbf{x}, \mathbf{w_r}) < d^2(\mathbf{x}, \mathbf{w_0}) = d^2(\mathbf{x}, \mathbf{z})$$

This contradicts our assumption that $\mathbf{z}$ is the point in B that is closest to $\mathbf{x}$, thus equation (3.2) must hold for all $\mathbf{y} \in B$. ∎

Before we move on to the next lemma, in order to reduce the notational complexity, we first define the *convex hull* of a set of N points:

Definition In $\mathbb{R}^n$, the *convex hull* of N points $P = \{p_1, p_2, \cdots, p_N\}$ is given by

$$Conv\{p_1, p_2, \cdots, p_N\} = \{\sum_{j=1}^{N} \lambda_j p_j \mid \lambda_j \geq 0 \text{ for all } j, \text{ and} \sum_{j=1}^{N} \lambda_j = 1\}$$

Equivalently, the convex hull of the set of points P is the intersection of all convex sets containing P.

From Lemma 3.3.2, the following can be easily proved:

Lemma 3.3.3 *If $A = (a_{ij})$ is an $m \times n$ matrix, then either (I) or (II) must hold:*

(I). The point 0 is contained in the convex hull of the $m + n$ points:

$$\begin{aligned} a_1 &= (a_{11}, a_{21}, \cdots, a_{m1}) \\ &\vdots \\ a_n &= (a_{1n}, a_{2n}, \cdots, a_{mn}) \\ e_1 &= (1, 0, \cdots, 0) \\ &\vdots \\ e_m &= (0, 0, \cdots, 1) \end{aligned}$$

(II). There exists numbers $x_1, x_2, \cdots, x_m$ such that

$$\begin{cases} x_i > 0 \\ \sum_{i=1}^{m} x_i = 1 \\ \sum_{i=1}^{m} a_{ij}x_i > 0 \qquad for \quad j = 1, 2, \cdots, n \end{cases}$$

Now we are ready to prove the Minimax Theorem:

Proof By Lemma 3.3.3, either of the following has to hold:

(i). If (I) holds, then 0 is in the convex hull of the $m + n$ vectors. Thus, 0 can be written as a convex linear combination of $a_1, a_2, \cdots, a_n, e_1, \cdots, e_m$, i.e. there exists real numbers $s_1, s_2, \cdots, s_{m+n}$, such that

$$\sum_{j=1}^{n} s_j a_{ij} + s_{n+i} = 0 \quad \text{for} \quad i = 1, 2, \cdots, m$$

where $s_j \geq 0$ for all j, and $\sum_{j=1}^{m+n} s_j = 1$.

Since $e_1, e_2, \cdots, e_m$ are linearly independent, we cannot have $s_1 = s_2 = \cdots = s_n = 0$ (otherwise, 0 is a convex linear combination of $e_1, e_2, \cdots, e_m$). Thus, $s_j > 0$ for some $j \in 1, 2, \cdots, n$, and we know $\sum_{j=1}^{n} s_j > 0$.

Let $u_j = s_j / \sum_{j=1}^{n} s_j$, $j = 1, 2, \cdots, n$, and we have

$$\begin{cases} u_j \geq 0 \\ \sum_{j=1}^{n} u_j = 1 \\ \sum_{j=1}^{n} a_{ij}u_j = -s_{n+i} / \sum_{j=1}^{n} s_j \leq 0 \quad i = 1, 2, \cdots, m. \end{cases}$$

Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$. Therefore, $v_2(\mathbf{u}) \leq 0$ (because for any pure strategies player 1 chooses, the payoff to 2 playing strategy $\mathbf{u}$ is always non-positive; so any mixed strategies of player 1 will give the same results.) Thus, in this case, the minimax value for player 2 satisfies $v_{II} \leq 0$.

(ii). If (II) in Lemma 3.3.3 holds, then let $\mathbf{x} = (x_1, x_2, \cdots, x_m)$. We have $v_1(\mathbf{x}) > 0$, so in this case, the maximin value for player 1 satisfies $v_I > 0$.

Now by Lemma 3.3.3, we know that either (i) or (ii) has to be true. Therefore, we *cannot* have $v_I \leq 0 < v_{II}$. Suppose now we transform this game, called A, to game B by adding a scalar k ($k \in \mathbb{R}$) to all entries a_{ij}:

$$b_{ij} = k + a_{ij}$$

It is obvious that the minimax value for player 2 and maximin value for player 1 both increase by k. Now suppose we have $v_I(A) < v_{II}(A)$, then it is possible to choose $k \in \mathbb{R}$, such that

$$v_I(A) + k \leq 0 < v_{II}(A) + k,$$

which means that $v_I(B) \leq 0 < v_{II}(B)$, but we have already shown by (i) and (ii) that this is impossible. Thus, $v_I(A) \geq v_{II}(A)$. Since A is arbitrarily chosen, then $v_I \geq v_{II}$ is true for any game. By Lemma 3.3.1, we have $v_I \leq v_{II}$. Thus $v_I = v_{II}$, and we proved the claim. ∎

In an ingenious proof of the existence of what is now known as the *Nash Equilibrium* for non-cooperative games, John Nash showed that the Minimax Theorem is actually a special case of the exitence theorem for 2-person zero-sum games. The more general case for all finite non-cooperative games will be presented in the next chapter.

Chapter 4

Non-Cooperative Games

In this chapter, we consider both zero-sum and non-zero-sum non-cooperative games. Recall that in non-zero-sum non-cooperative games, no pre-play communication is allowed among players. Therefore, the complication of coalition formation and contract negotiation will not come until the next chapter. The notation from the zero-sum cases are still applicable to all non-cooperative games.

4.1 Nash Equilibrium

We first define Nash Equilibrium of a non-cooperative game.

Let α_{ij} $(j = 1, 2, \cdots, m)$ be a pure strategy for player i, and let $s_i = (x_{i1}\alpha_{i1}, x_{i2}\alpha_{i2}, \cdots, x_{im}\alpha_{im})$ where $x_{ij} \geq 0$ and $\sum_{j=1}^{m} x_{ij} = 1$, so s_i represents a mixed strategy for player i.

Let $\mathbf{s} = (s_1, s_2, \cdots, s_n)$. We think of $\mathbf{s}$ as a point in the product space of the vector spaces containing the mixed strategies of all n players. Let $p_i(\mathbf{s})$ be the payoff function for player i. It is a mapping from the set of all n-tuples of mixed strategies into $\mathbb{R}$.

Also we introduce the notation of $(\mathbf{s}, t_i)$:

$$(\mathbf{s}, t_i) = (s_1, s_2, \cdots, s_{i-1}, t_i, s_{i+1}, \cdots, s_n)$$

So $(\mathbf{s}, t_i)$ is a modification of $\mathbf{s}$, where player i employs a strategy t_i instead of s_i, and the strategies of all the other players remain the same.

Definition Suppose in an n-person game, S_i is the set of strategies available to player i. A *Nash equilibrium* of the game is an n-tuple $\mathbf{s}$, if and only if

$$p_i(\mathbf{s}) = \max_{r_i \in S_i} p_i(\mathbf{s}, r_i).$$

In other words, Nash equilibrium means that no player has any incentive to change his own equilibrium strategy, *given* that none of the other players are going to change strategies. If the assumptions are correct, then the game is stable at the Nash equilibrium point.

Next we state Brouwer's Fixed-point Theorem, based on which Nash proved the existence of Nash Equilibrium for all finite non-cooperative games. A proof of Brouwer's Theorem can be found in [8].

Theorem 4.1.1 Brouwer's Fixed-point Theorem
Suppose $\mathbb{S}^n \subseteq \mathbb{R}^n$ is a compact convex set. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous

function. Then there exists $x \in \mathbb{S}^n$ such that

$$f(x) = x$$

In his proof [21], Nash defined a set of continuous functions of $\mathbf{s}$ by

$$\phi_{i\alpha}(\mathbf{s}) = \max(0, p_{i\alpha}(\mathbf{s}) - p_i(\mathbf{s}))$$

where $p_{i\alpha}(\mathbf{s}) = p_i(\mathbf{s}, \alpha_{ij})$.

For each component s_i of $\mathbf{s}$, a modification s_i' is defined by

$$s_i' = \frac{s_i + \sum_\alpha \phi_{i\alpha}(\mathbf{s})\alpha_{ij}}{1 + \sum_\alpha \phi_{i\alpha}(\mathbf{s})}$$

Let $\mathbf{s}' = (s_1', s_2', \cdots, s_n')$. A mapping $T : \mathbf{s} \to \mathbf{s}'$ is therefore constructed. Nash showed that the fixed points of the mapping are the equilibrium points. The central idea in his proof is that an equilibrium point $\mathbf{s}$ does not use α_{ij} (i.e. the mixed strategy s_i does not use α_{ij}), unless α_{ij} is an optimal pure strategy for player i.

Using Brouwer's Fixed-point Theorem, Nash proved the existence of a fixed point under T, and thus the existence of the well-known *Nash equilibrium* for non-cooperative games:

Theorem 4.1.2 *Every finite non-cooperative game has an equilibrium point.*

The appeal of Nash Equilibrium lies in its intuitiveness and its corresponding existence theorem for all non-cooperative games, which we will illustrate further with some examples in the next section.

4.2 Examples of Nash Equilibrium in Two-person Non-cooperative Games

In this section, we give examples of some famous 2-person games, and examine some of their implications in the real world.

(I). Prisoner's Dilemma.

One of the most famous non-cooperative games, Prisoner's Dilemma is attributed to A.W. Tucker and has received considerable public attention. The payoff matrix associated with the game is as follows, where a_i, b_i denote strategies for player i $(i = 1, 2)$.

	a_2	b_2
a_1	(-5,-5)	(0,-6)
b_1	(-6,0)	(-1,-1)

The interpretation of the game is that two suspects of a crime were arrested and interrogated separately: for both, strategy a denotes "confess" and strategy b denotes "not confess." The absolute value of the payoff denotes the number of years each prisoner will be sentenced. Clearly, no matter what the other person's choice is, for each prisoner, "confess" is a dominant strategy over "not confess." The Nash equilibrium in this case is the pure strategy pair $(-5, -5)$, which collectively speaking is the worst outcome for both players.

We observe that in this case, the Nash equilibrium has nothing to do with social efficiency. In fact, Prisoner's Dilemma is commonly used to model environmental issues concerning negative externality, as well as pricing strategy of firms in a duopoly (oligopoly) market. The solution can be modified if the game is played repeatedly (thus cooperative behavior will be rewarded), or by allowing pre-play communication such that players can make a binding agreement, usually under the supervision of a potent third party (a mafia in the case of the prisoners, government regulation in the case of negative externality, and a cartel in the case of an oligopoly market).

(II). Battle of the Sexes and Chicken.

The *Battle of the Sexes* game got its name from the story in which the husband and wife are deciding independently where to go on a Saturday night. While the husband prefers a football match (strategy a) to a movie (strategy b), and vice versa for the wife, both find it more important to have each other's company. Thus, in this game, there are two pure strategy Nash equilibria (a_1, a_2) and (b_1, b_2), which occur when the choices of the couple are aligned.

	a_2	b_2
a_1	(3,1)	(0,0)
b_1	(-1,-1)	(1,3)

The conundrum arising from the non-uniqueness of Nash equilibrium is that the couple still cannot decide which equilibrium point to choose, and it is possible that if each person chooses altruistically, they end up at the worst-case scenario $(-1, -1)$. One alternative is to choose a mixed strategy Nash equilibrium such that one's security level is maximized. In this case, player 1, the husband, employs mixed strategy $(0.8a_1, 0.2b_1)$ and player 2, the wife, chooses $(0.2a_2, 0.8b_2)$. Since the game is symmetric, both players maximize their security level at 0.6. Another alternative for the Battle of the Sexes games is to find a *focal point*, an equilibrium that has some property that distinguishes it from all the other equilibria. The focal point could usually be derived from either a social, cultural norm or family tradition.

Similar to the Battle of the Sexes game, a *Chicken* game also gives two pure-strategy Nash equilibria. The difference is that in this case, the equilibria occur when the choices of the two players are not aligned.

	a_2	b_2
a_1	(-5,-5)	(5,-1)
b_1	(-1,5)	(-1,-1)

The game got its name from a popular activity in which two adolescents drive a motorcycle towards each other, and the first to swerve loses and is humiliated as the "chicken." However, if nobody swerves, then serious physical damage is likely to occur, giving both players more negative payoffs. The Nash equilibria are (a_1, b_2), $(b_1,\ a_2)$ and $(0.6a_1 + 0.4b_1, 0.6a_2 + 0.4b_2)$. Involved parties can send convincing signals to show their commitment not to "swerve," by, for example, ostentatiously disabling the steering wheel before the game. In the realm of international politics, nuclear brinkmanship is often modeled by a Chicken game.

(III). Sports game.

Compared with the Battle of the Sexes and Chicken games, in which contention follows from the multiplicity of pure strategy Nash equilibria, a sports game has no pure strategy equilibrium points.

	a_2	b_2
a_1	(0,4)	(4,0)
b_1	(4,0)	(0,4)

In tennis, for example, player 1 can hit short (a_1) or long (b_1), while player 2 can go to the net (a_2) or stay at the bottom line (b_2). A simplified payoff matrix for a tennis match is as shown above. It is easy to observe that there is no pure strategy equilibrium — otherwise, mathematicians can all be great tennis players. The mixed-strategy Nash equilibrium is when both players employ their two strategies with equal probability.

As we have illustrated with the four types of games, to find a pure-strategy Nash equilibrium, one just needs to do a run-through check of all the outcomes to find the intersections of the minimax/maximin strategies. A mixed-strategy Nash equilibrium, on the other hand, requires that a player get the same payoff from any of his pure strategies with positive probability in his mixed strategy. Thus, we can equate the expected payoffs from a player's different pure strategies to find the probability measures of a mixed strategy Nash equilibrium.

In recent years, with the increasing power and speed of computers, linear programming has enjoyed greater popularity as a means to solve optimization problems. By transforming a game into sets of inequalities, we can use linear programming to solve for Nash equilibria in n-person non-cooperative games. See Vanderbei [34] for a detailed account of linear programming techniques and applications of the Simplex algorithm.

Chapter 5

Cooperative Games

The study of cooperative games, in contrast to non-cooperative games, entails much more diverse solution concepts. With pre-play communication allowed, players can negotiate with each other and use various bargaining strategies, such as sending signals, making credible threats and forming coalitions, which consequently complicates the analysis of cooperative games. Different solutions arise based on different assumptions, and each has its own applicability and caveats. In this chapter, we will examine in detail and critically analyze a number of solution concepts.

Let's first explain explicitly the assumptions we make for *cooperative games*:

(i). All agreements reached by pre-play communication are binding and enforceable. Therefore, coalitions formed after bargaining are not to be broken during the games.

(ii). A player's evaluation of the outcomes of the game is not disturbed by the pre-play communication. Therefore, the payoff matrix is unaltered by these bargaining and negotiations.

In this chapter, we first examine the Nash bargaining solution in some detail, then move on to an elaborated discussion of the "bargaining set," "kernel," "core," and "nucleolus." We will also briefly discuss other solution concepts, such as the Von Neumann and Morganstern solution, *ψ-stability*, and the *Shapley value*. In essence, each solution rests on a number of assumptions or requirements that are set up to reduce the number of possible solutions to cooperative games. We will then provide some intuitive interpretations of these solutions, evaluate their assumptions, and compare and contrast them using some simple examples.

5.1 Nash Bargaining Solution

5.1.1 Notation and Existence Theorem

Suppose in a two-person bargaining game, F is the feasible set of points that the solutions can be chosen from. It is reasonable to assume that F should be a compact set, since in most cases the feasible points for both players are bounded and the extreme cases are always achievable. We also assume that F is a convex subset of $\mathbb{R}^2$. The convexity can be explained by allowing players to apply randomized strategies: for example, if two feasible payoffs for player 1 are x_1 and y_1, then $px_1 + (1-p)y_1$ where $0 \leq p \leq 1$ can be interpreted as getting payoff x_1 with a probability of p, and y_1 with a probability of $1-p$, which is also feasible.

Let $\mathbf{v} = (v_1, v_2)$, where v_1 and v_2 are current payoffs to player 1 and player 2, respectively. Thus, if no bargaining takes place or no other agreement reached, $\mathbf{v} \in \mathbb{R}^2$ is called the *status quo point*.

We assume

$$F \cap \{(x_1, x_2) | x_1 \geq v_1 \quad \text{and} \quad x_2 \geq v_2\} \neq \phi;$$

thus there is room for bargaining that can be beneficial to both players.

Definition A bargaining problem is an *essential bargaining problem* if there exists $\mathbf{y} = (y_1, y_2) \in F$ such that

$$y_1 > v_1 \quad \text{and} \quad y_2 > v_2.$$

Otherwise, the problem is called *inessential*.

In other words, there exists an allocation in the feasible set that is strictly better than the status quo point for both players. If no such allocation $\mathbf{y}$ exists, then at least one player's payoff is already maximized, so there is no need for bargaining.

The feasible set F and the status quo point $\mathbf{v}$ can determine a bargaining game, which we denote by $(F,\ \mathbf{v})$. We denote the solution function by $\Phi(F, \mathbf{v}) = (\phi_1(F, \mathbf{v}), \phi_2(F, \mathbf{v}))$, where $\phi_i(F, \mathbf{v})$ is the payoff to player i $(i = 1, 2)$. If $\phi_1(F, \mathbf{v}) > \phi_1'(F, \mathbf{v})$ and $\phi_2(F, \mathbf{v}) > \phi_2'(F, \mathbf{v})$, we say $\Phi(F, \mathbf{v}) > \Phi'(F, \mathbf{v})$.

Nash proposed five axioms that a reasonable bargaining solution must satisfy:

1. Pareto efficiency:

   If $\mathbf{x} = (x_1, x_2) \in F$ and $\mathbf{x} \geq \Phi(F, \mathbf{v})$, then $\mathbf{x} = \Phi(F, \mathbf{v})$.

   This axiom requires that there is no allocation that would give both players strictly better payoffs than the bargaining solution.

2. Individual rationality:

   $\Phi(F, \mathbf{v}) \geq \mathbf{v}$. We assume each player wants to maximize his payoff: no player will bargain to get a payoff less than what he gets from his status quo point.

3. Scale covariance:

   Assume $\Phi(F, \mathbf{v})$ is the solution to a bargaining problem with status quo point $\mathbf{v} = (v_1, v_2)$. Choose $\lambda_1, \lambda_2, \gamma_1, \gamma_2 \in \mathbb{R}$, with $\lambda_1, \lambda_2 > 0$. Let

   $G = \{(\lambda_1 x_1 + \gamma_1, \lambda_2 x_2 + \gamma_2) \mid (x_1, x_2) \in F\}$

   $\mathbf{w} = (\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2)$.

   Then, the solution to the bargaining game $(G, \mathbf{w})$ is

   $\Phi(G, \mathbf{w}) = (\lambda_1 \phi_1(F, \mathbf{v}) + \gamma_1, \lambda_2 \phi_2(F, \mathbf{v}) + \gamma_2)$

   Thus the solution function preserves an affine transformation.

4. Independence of irrelevant alternatives:

   If $G \subseteq F$, and $\Phi(F, \mathbf{v}) \in G$, then $\Phi(G, \mathbf{v}) = \Phi(F, \mathbf{v})$.

   This axiom is intuitively reasonable: taking out some feasible points that are not in the solution set will not affect the bargaining solution.

5. Symmetry:

   If $v_1 = v_2$ and $\{(x_2, x_1) \mid (x_1, x_2) \in F\} = F$

   then $\phi_1(F, \mathbf{v}) = \phi_2(F, \mathbf{v})$

   Thus the labeling of the players doesn't make any concrete difference.

We can see that each of the above axioms is by itself a reasonable requirement for an ideal solution function. Surprisingly, Nash [22] proved that there exists a unique function that satisfies all the five axioms taken together.

Theorem 5.1.1 *There exists a unique function* $\Phi : (F, \mathbf{v}) \to (x_1, x_2) \in F$ *that satisfies the five axioms above. For every two-person bargaining problem* $(F, \mathbf{v})$*, the solution function is defined as*

$$\begin{aligned} \Phi(F, \mathbf{v}) &= (x_1, x_2), \\ \textit{where } \; f(x_1, x_2) &= (x_1 - v_1)(x_2 - v_2) \;\; \textit{is maximized, for } \; \mathbf{x} \geq \mathbf{v}, \mathbf{x} \in \mathbf{F}. \end{aligned}$$

Such a Φ is also called a **Nash bargaining solution**.

5.1.2 Proof of the Uniqueness of Nash Bargaining Solution

We first provide our proof of the uniqueness of a maximum for $f(x_1, x_2)$ in an essential game, based on which we then illustrate Myerson's (see [19, page 379]) elegant proof of the uniqueness of the Nash Bargaining solution.

Lemma 5.1.1 *In an essential game* $(F, \mathbf{v})$*, there exists a unique vector* $\mathbf{x} = (x_1, x_2)$ *where* $\mathbf{x} \in F$ *and* $\mathbf{x} > \mathbf{v}$*, such that* $f(x_1, x_2) = (x_1 - v_1)(x_2 - v_2)$ *is maximized.*

Proof We prove by contradiction.

Suppose we have $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}' = (x'_1, x'_2)$, where $\mathbf{x} \neq \mathbf{x}'$ and $f(x_1, x_2) = f(x'_1, x'_2) \geq f(y_1, y_2)$ for all $\mathbf{y} = (y_1, y_2) \in F$ and $\mathbf{y} > \mathbf{v}$.

Consider the midpoint $\mathbf{z} = (\frac{x_1+x'_1}{2}, \frac{x_2+x'_2}{2})$ between $\mathbf{x}$ and $\mathbf{x}'$. Since F is a convex set, we know that $\mathbf{z} \in F$. From $\mathbf{x} > \mathbf{v}$ and $\mathbf{x}' > \mathbf{v}$, it follows that $\mathbf{z} > \mathbf{v}$. Therefore,

$$f(x_1, x_2) = f(x'_1, x'_2) \geq f(\frac{x_1 + x'_1}{2}, \frac{x_2 + x'_2}{2}) \tag{5.1}$$

Since we have

$$\begin{aligned} (x_1 - v_1)(x_2 - v_2) &= (x'_1 - v_1)(x'_2 - v_2) \\ x_i &> v_i \quad \text{for } i = 1, 2 \\ x'_i &> v_i \quad \text{for } i = 1, 2 \end{aligned}$$

and we know $\mathbf{x} \neq \mathbf{x}'$, by symmetry, we can assume $x_1 > x'_1$ without loss of generality. So there are two cases:

- If $x_2 \geq x'_2$, then $f(x_1, x_2) > f(x'_1, x'_2)$, which contradicts our assumption.

- If $x_2 < x'_2$, then

$$(x_1 - x'_1)(x_2 - x'_2) < 0 \tag{5.2}$$

We know from equation (5.1) that

$$\begin{aligned}
2f(\frac{x_1+x_1'}{2},\frac{x_2+x_2'}{2}) &\leq f(x_1,x_2)+f(x_1',x_2') \\
2(\frac{x_1+x_1'}{2}-v_1)(\frac{x_2+x_2'}{2}-v_2) &\leq (x_1-v_1)(x_2-v_2)+(x_1'-v_1)(x_2'-v_2) \\
(x_1+x_1')(x_2+x_2') &\leq 2(x_1x_2+x_1'x_2') \\
x_1x_2'+x_1'x_2 &\leq x_1x_2+x_1'x_2' \\
(x_1-x_1')(x_2-x_2') &\geq 0
\end{aligned}$$

which is contrary to equation (5.2). Thus, we proved by contradiction that there there is a unique vector $\mathbf{x} \in F, \mathbf{x} > \mathbf{v}$ such that $f(x_1, x_2)$ is maximized.

■

Now we can prove the uniqueness of the Nash bargaining solution.

Proof We first examine the case in which $(F, \mathbf{v})$ is an essential bargaining problem. Let $\mathbf{x}$ be the unique point in F such that $f(x_1, x_2)$ is maximized.

Let $\lambda_i = 1/(x_i - v_i) > 0$ and $\gamma_i = -v_i/(x_i - v_i)$, for $i = 1, 2$. Define a function $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$L(\mathbf{y}) = (\lambda_1 y_1 + \gamma_1, \lambda_2 y_2 + \gamma_2).$$

Let $G = \{L(\mathbf{y}) \mid \mathbf{y} \in F\}$. Therefore, G is an affine transformation of F, so it is also a convex set. Let $\mathbf{z} = L(\mathbf{y})$, then we find for any $\mathbf{y} \in F$,

$$\begin{aligned} z_1 z_2 &= (\lambda_1 y_1 + \gamma_1)(\lambda_2 y_2 + \gamma_2) \\ &= \lambda_1 \lambda_2 (y_1 - v_1)(y_2 - v_2) \\ &= \lambda_1 \lambda_2 f(y_1, y_2) \end{aligned}$$

Since $\mathbf{x}$ maximizes $f(x_1, x_2)$, so $z_1 z_2$ is maximized when $\mathbf{z} = L(\mathbf{x})$. By some simple algebra, we know that $L(\mathbf{x}) = (1, 1)$. Now the hyperbola $z_1 z_2 = 1$ has a slope of -1 at the point $(1, 1)$. Therefore, we can find the equation of the tangent line to $z_1 z_2 = 1$ at $(1, 1)$: $z_1 + z_2 = 2$. Since G is convex, and $(1, 1)$ is the unique point in G that intersects $z_1 z_2 = 1$, it follows that G is on one side of $z_1 + z_2 = 2$. Since $z_1 z_2$ is maximized at $(1, 1)$, so G has to be a subset of $E = \{(z_1, z_2) \mid z_1 + z_2 \leq 2\}$.

By axiom 1 and 5, we have $\Phi(E, (0, 0)) = (1, 1)$.
By axiom 4, we have $\Phi(G, (0, 0)) = (1, 1)$.
By axiom 3, we know, for $\phi_i = \phi_i(F, \mathbf{v})$

$$\Phi(G, (0, 0)) = (\frac{\phi_1}{x_1 - v_1} - \frac{v_1}{x_1 - v_1}, \frac{\phi_2}{x_2 - v_2} - \frac{v_2}{x_2 - v_2}) = (1, 1).$$

Thus we have $\phi_1 = x_1$ and $\phi_2 = x_2$, which means that $\Phi(F, \mathbf{v}) = (x_1, x_2)$ (so axiom 2 is automatically satisfied). Therefore, we proved that the only function Φ that satisfies the five axioms selects the allocation $\mathbf{x}$ that maximizes $f(x_1, x_2)$. By the uniqueness of $\mathbf{x}$, Φ is a unique solution to the essential bargaining problem.

For the inessential case, the argument is fairly straight forward. First, we show that if $(F, \mathbf{v})$ is inessential, then either $x_1 - v_1 = 0$ or $x_2 - v_2 = 0$ has to be true for all $\mathbf{x} \geq \mathbf{v}$. Because otherwise, if we have $y_1 > v_1$, $z_2 > v_2$, and $\mathbf{y} \geq \mathbf{v}$, $\mathbf{z} \geq \mathbf{v}$, then $(1/2)\mathbf{y} + (1/2)\mathbf{z} > \mathbf{v}$, which contradicts that $(F, \mathbf{v})$ is inessential.

Without loss of generality, let us suppose $x_1 = v_1$ for all $\mathbf{x} \geq \mathbf{v}$. By assumption that F is compact, we simply choose

$$\mathbf{x} = (v_1, x_2), \quad \text{where} x_2 = \max\{z_2 : (v_1, z_2) \in F\}.$$

Thus, this $\mathbf{x}$ is the only point that satisfies axioms 1 and 2. So Φ is uniquely determined in the inessential bargaining case as well. In this case, $f(x_1, x_2) = 0$ for all the individually rational allocations in the game. ∎

The existence theorem is rather easy to prove. We simply verify that $\Phi(F, \mathbf{v})$ as defined in Nash bargaining solution satisfies the five axioms. It is straight forward, so we will not include it.

In assessment of the Nash bargaining solution, axiom 4, the independence of irrelevant alternatives, has been much criticized. Suppose, for example, the Nash bargaining solution is (x_1, x_2) for a game $(F, \mathbf{v})$. If we truncate all the feasible sets $H = \{\mathbf{y} \in F \mid y_1 > x_1\}$, according to the axiom, the solution doesn't change. However, we observe that in the revised game $(H, \mathbf{v})$, player 1 is getting his highest payoff possible even though all his better options have been eliminated. This surely appears very "unreasonable" to player 2. Luce and Raiffa (see [11, page 133]) gave a very neat argument with graphic interpretation.

5.1.3 Interpersonal Utility Comparison and Transferable Utility

In Chapter 2, we mentioned that the comparison of interpersonal utility raises the question of which utility scale for the players should be employed. In the case of the Nash bargaining solution, two principles we consider here are "equity" and "efficiency." The following theorem shows that the Nash bargaining solution turns out to be a nice synthesis of the two.

Theorem 5.1.2 *Let $(F, \mathbf{v})$ be an essential two-person bargaining problem, and let $\mathbf{x}$ be an allocation vector such that $\mathbf{x} \in F$ and $\mathbf{x} \geq \mathbf{v}$. Then $\mathbf{x}$ is the Nash bargaining solution to the game $(F, \mathbf{v})$, if and only if there exist positive numbers λ_1 and λ_2, such that*

$$\begin{aligned} \lambda_1 x_1 - \lambda_1 v_1 &= \lambda_2 x_2 - \lambda_2 v_2 \\ \lambda_1 x_1 + \lambda_2 x_2 &= \max_{\mathbf{y} \in F} (\lambda_1 y_1 + \lambda_2 y_2) \end{aligned}$$

A detailed proof can be found in Myerson (see [19, page 383]). Intuitively, we can see that λ_1 and λ_2 are introduced in the utility scale for the interpersonal comparison between player 1 and player 2. The first equation means that the λ-weighted benefit that player 1 obtains by moving from the status quo point to the bargaining solution is the same as the benefit to player 2 from such a move (the equity principle). The second equation is a very nice property of the bargaining solution: the λ-weighted sum of the payoff to the players is maximized, so it is efficient (the efficiency principle). We call λ_1 and λ_2 *natural scale factors* for the game $(F, \mathbf{v})$.

The theorem states that the Nash bargaining solution is both "fair" and "efficient," which are two of the most important notions in many social and economic context — a very appealing property of the Nash bargaining solution. Notice that the efficiency condition here indicates that the solution is socially efficient, which is not the same as the Pareto efficiency requirement in axiom 1 — the former stresses on maximizing the collective payoff to both players, whereas the latter is about individual payoffs. This property of Nash bargaining solution strikes us with its resemblance to the "invisible hand" argument of Adam Smith, the founding concept of neoclassical economics, i.e. what is the best for each individual will collectively be the best for the society.

Now let's assume that the players can not only represent their utility in a common unit, say in monetary terms, so that interpersonal comparison is possible (λ-weighted), they can also give any amount of money to any other player. Thus, we say that $(F, \mathbf{v})$ is a game with *transferable utility*.

Let F be the feasible set in a game with transferable utility, and let v_{12} represent the maximum transferable wealth that the players can jointly achieve. Let S_i be the set of strategies available to player i $(i = 1, 2)$. Thus,

$$
\begin{aligned}
F &= \{\mathbf{y} \in \mathbb{R}^2 \mid y_1 + y_2 \leq v_{12}\}, \quad \text{where} \\
v_{12} &= \max_{\mu \in S_i} (x_1(\mu) + x_2(\mu))
\end{aligned}
$$

In games with transferable utility, one unit of transferred utility weights the same for both players, therefore $\lambda_1 = \lambda_2$. We can simplify the condition for $\phi(F, \mathbf{v})$ from Theorem 5.1.2:

$$\begin{aligned} \phi_1(F, \mathbf{v}) - v_1 &= \phi_2(F, \mathbf{v}) - v_2 \\ \phi_1(F, \mathbf{v}) + \phi_2(F, \mathbf{v}) &= v_{12} \end{aligned}$$

which gives us the formula for the Nash bargaining solution for a 2-person game with transferable utility:

$$\phi_1(F, \mathbf{v}) = (v_{12} + v_1 - v_2)/2 \tag{5.3}$$

$$\phi_2(F, \mathbf{v}) = (v_{12} + v_2 - v_1)/2 \tag{5.4}$$

5.1.4 Determination of the Status Quo Point

Up to this point, we have assumed the status quo point $\mathbf{v} = (v_1, v_2)$ is given *a priori*. In real-world games, however, we know that before two or more players sit down at the negotiation table, there is often no such “status quo point” that will be accepted by all parties in case the bargaining fails. Also, as we can see in the preceding formula for a 2-person game with transferable utility, the final Nash bargaining payoff to player 1 not only depends on his own initial status quo point v_1, it also depends on the status quo point of his opponent v_2. Therefore, it is in the interest of the players to secure a better status quo point before the real bargaining takes place. In these circumstances, there are three possible choices of a status quo point, which we illustrate with the following example of a game with transferable utility.

Consider the following 2-person game:

	a_2	b_2
a_1	$(5,3)$	$(-1,2)$
b_1	$(2,-1)$	$(1,5)$

(1). Minimax value

Each player assumes that the other player(s) are playing purely competitively, as they would in a zero-sum game. Therefore, the players play most offensively to drive down the payoff to their opponent as much as possible, so as to guarantee a maximized security level.

In the above example, the game that player 1 will be playing essentially, assuming offensive strategy of his opponent, is the following zero-sum game:

	a_2	b_2
a_1	$(5,-5)$	$(-1,1)$
b_1	$(2,-2)$	$(1,-1)$

There is a pure strategy equilibrium (b_1, b_2). Thus, the minimax value for player 1 is 1.

Similarly, player 2 is essentially playing the following zero-sum game:

	a_2	b_2
a_1	$(-3,3)$	$(-2,2)$
b_1	$(1,-1)$	$(-5,5)$

This game doesn't have a pure strategy equilibrium, instead, the solution is a mixed strategy for both players: $(6/7a_1 + 1/7b_1, 3/7a_2 + 4/7b_2)$. Thus, the minimax value for player 2 is 17/7.

Therefore, the status quo point will be $\mathbf{v_1} = (1, 17/7)$. The maximum total payoff available to the players is 8 in this game. We calculate the Nash bargaining solution and get $\Phi(F, \mathbf{v_1}) = (23/7, 33/7)$.

(2). Defensive equilibrium

Each player assumes that the other player(s) are playing purely cooperatively. In other words, each player is only interested in his own individual payoff, and chooses his strategy accordingly.

With the above example, we know that both $(5, 3)$ and $(1, 5)$ are Nash equilibria. There is also a mixed strategy equilibrium for the non-zero-sum game: $(6/7a_1 + 1/7b_1, 2/5a_2 + 3/5b_2)$. If we use the mixed defensive equilibrium, the status quo point will be $\mathbf{v_2} = (7/5, 17/7)$, and the Nash bargaining solution is $\Phi(F, \mathbf{v_2}) = (122/35, 158/35)$.

(3). Threat game solution

We have made the argument that the determination of the status quo point does have an impact on the final solution. Here we formalize this incentive for players to act competitively to get a more favorable status quo point by introducing a rational threat game.

In a threat game, each player can leverage his ability to drive down the payoff of the opponent by making threats in order to reach an advantageous pre-bargaining position. Let S_i be the set of strategies available to player i $(i = 1, 2)$. Suppose that before the negotiation begins, each player i chooses a threat strategy $\tau_i \in S_i$. We assume that if the players fail to reach an agreement during the bargaining, they are committed to carry out their respective threat. Therefore, the status quo point $\mathbf{v}$ is a function of the threats: $\mathbf{v} = (v_1(\tau_1, \tau_2), v_2(\tau_1, \tau_2))$. Denote by $w_i(\tau_1, \tau_2)$ the payoff that player i gets in the Nash bargaining solution with status quo point $\mathbf{v}$, thus

$$w_i(\tau_1, \tau_2) = \phi_i[F,\ (v_1(\tau_1, \tau_2), v_2(\tau_1, \tau_2))\,].$$

The ultimate purpose of making a rational threat is to maximize the payoff in the Nash bargaining solution, rather than actually carrying out the threat. Therefore, we define a *rational threat* as follows:

Definition Suppose S_i is the set of strategies available to player i. The pair (τ_1, τ_2) is a *rational threat* if and only if

$$\begin{aligned} w_1(\tau_1, \tau_2) &\geq w_1(\sigma_1, \tau_2),\ \forall \sigma_1 \in S_1, \\ w_2(\tau_1, \tau_2) &\geq w_2(\tau_1, \sigma_2),\ \forall \sigma_2 \in S_2. \end{aligned}$$

Notice the similarities of the above definition to the existence of a Nash Equilibrium in the non-cooperative case. Thus the existence of a rational threat can also be proved by using the fixed-point theorem.

Myerson [19, Chapter 8] showed that for a game with transferable utility, the payoffs in the threat game are given by equations (5.3) and (5.4) in the preceding section:

$$\begin{aligned} w_1(\tau_1, \tau_2) &= \frac{v_{12} + v_1(\tau_1, \tau_2) - v_2(\tau_1, \tau_2)}{2} \\ w_2(\tau_1, \tau_2) &= \frac{v_{12} + v_2(\tau_1, \tau_2) - v_1(\tau_1, \tau_2)}{2} \end{aligned}$$

Thus, we transform the original game in this example into its rational threat game:

	a_2	b_2
a_1	$(5, 3)$	$(2.5, 5.5)$
b_1	$(5.5, 2.5)$	(2, 6)

This is a zero-sum game, and the unique equilibrium is $(2.5, 5.5)$. Thus the equilibrium of the threat game is (a_1, b_2), so the status quo point is $v_3 = (-1, 2)$, and the Nash bargaining solution with rational threats is $\Phi(F, \mathbf{v_3}) = (2.5, 5.5)$.

In essence, the minimax value and the defensive equilibrium theory of the status quo point differ on their assumption of opponent behavior. While the minimax value assumes that the opponent is antagonistic and aligns his interest in direct opposite direction of yours, defensive equilibrium presumes that the opponent is generous, and only cares about his own individual payoff.

The third method, the threat game solution, synthesizes the defensive and offensive behavior into one zero-sum-game — for player 1, instead of trying to maximize v_1 (as in the case of defensive equilibrium) or to minimizes v_2 (as in the case of minimax value), he aims at maximizing $(v_{12} + v_1 - v_2)$ instead. This gives the player a chance to commit to a strategy which he agrees to carry out in the event of disagreement, and thereby increase his bargaining power. In the meantime, the fact that threats are not actually intended to be carried out keeps the strategies from being purely offensive. Notice that the Nash bargaining solution with rational threats depends entirely on the *relative* size of the payoffs $(v_1 - v_2)$ between the players, rather than each individual's absolute payoff.

5.2 Bargaining Set, Kernel, Core and Nucleolus

The 2-person Nash bargaining solution can be easily generalized into n-person games. However, it is not widely used for the analysis of cooperative games when n is greater than 2, since the Nash bargaining solution doesn't incorporate the possibility of coalition making among subsets of the players. Another set of solution concepts for cooperative games, which includes coalitional analysis that we will present in detail, is *the bargaining set, kernel, core and nucleolus*, initially proposed and studied by Aumann, Davis, Maschler and Peleg. We will first introduce the notation involved, and then examine some existence theorems, mainly for two solution concepts that we are particularly interested in, the bargaining set (two types) and the nucleolus. We will also provide some simple examples to further illustrate these solutions.

5.2.1 Notation

In an n-person game, a *coalition* S is a subset of the set of players $N = \{1, 2, \cdots, n\}$, who have decided to act in the game as a group. For now, we assume that any subset of N can form a coalition; later, other requirements of a permissible coalition will be applied. The aim of coalition S is to maximize the group payoff (the sum of the payoffs to the individual members in the group).

The *characteristic function*, v, is a mapping from the set of coalitions to the real numbers: $v : S \mapsto \mathbb{R}$. The amount $v(S)$ is the maximum payoff that S can guarantee itself, regardless of the strategies of the other players, and $v(S)$ is called *the value of coalition* S.

The characteristic function v is not without restrictions. For cooperative games, it must satisfy:

- $v(\emptyset) = 0$

  The value of an empty set is equal to zero.

- If R and S are two disjoint subsets of players,

  $$v(R \cup S) \geq v(R) + v(S)$$

  This requirement simply means if the union of disjoint sets R and S achieves less than the sum of what R and S can get respectively playing by themselves, then such a coalition $(R \cup S)$ cannot be formed.

Consider an n-person cooperative game with a characteristic function v. For simplicity, let us assume that the games are 0-normalized, i.e. for each coalition S, $v(S) \geq 0$; and for each individual player i, $v(i) = 0$.

A *coalition structure* $\mathcal{B} = B_1, B_2, \cdots, B_m$ is a partition of the players $N = \{1, 2, \cdots, n\}$ into m non-empty coalitions.

Definition A *payoff configuration* (p.c.) is defined as

$(\mathbf{x}; \mathcal{B}) = (x_1, x_2, \cdots, x_n; B_1, B_2, \cdots, B_m)$

where, for $i = 1, 2, \cdots, n$, x_i represents the payoff to each individual.
If $x_i \geq 0$ and $\sum_{i \in B_j} x_i = v(B_j)$ for $j = 1, 2, \cdots, m$, then we call it an *individually rational payoff configuration* (i.r.p.c.).
If we make a stronger assumption about the permissible coalitions such that:

$$\sum_{i \in B} x_i \geq v(B)$$

for each $B \subset B_j$, $j = 1, 2, \cdots, m$, then we call the outcome a *coalitionally rational payoff configuration* (c.r.p.c.).

The rationale behind an i.r.p.c. is that the payoff to any individual in a coalition has to be greater than or equal to what he can obtain by playing on his own, (in this case, earning a payoff of 0), otherwise he won't join the coalition. A c.r.p.c. recognizes that if some members in a coalition B_j can obtain more by forming a permissible coalition among themselves, B_j cannot be formed in the first place.

The notion of "fairness" has always been an important concept in social psychology. Many experiments have shown that when all things are equal, people tend to think an equal payoff amongst all members is a most reasonable outcome. In games where the players are not "equal," however, it is accepted that the stronger player should get more. For a player, one way to convince his partners of his strength is by showing that he can have other better alternatives. His partners, in the meantime, can disregard such threat, by pointing their counter alternatives that would keep their original payoff without his help. Thus, these "threats" and "counter threats" are bases for bargaining in a game, from which "objections" and "counter objections" are defined.

Definition Let $(\mathbf{x}; \mathcal{B})$ be an i.r.p.c. in a game, and r and s be two distinct players in a coalition $B_j \subset \mathcal{B}$. An *objection of* r *against* s is a vector $\hat{y}^C$ whose coordinates are $\{y_k \mid k \in C\}$, and for which

$r \in C$ and $s \notin C$,

$y_r > x_r$,

$y_k \geq x_k$ for all $k \in C$.

Verbally, in his objection, player r claims that without the help of s, he can get more by forming a coalition C with some other players, whose payoffs will be no less than what they can obtain in the original coalition structure.

Definition Let $(\mathbf{x}; \mathcal{B})$ be an i.r.p.c. in a game, and let $\hat{y}^C$ be an objection of r against s, where $r, s \in B_j$. A *counter objection of* s *against* r is a vector $\hat{z}^D$ whose coordinates are $\{z_k \mid k \in D\}$, and for which

$s \in D$ and $r \notin D$,

$z_k \geq x_k$, for all $k \in D$,

$z_k \geq y_k$, for all $k \in C \cap D$.

Verbally, in his counter objection, player s claims to form a coalition D excluding r, in which all players get at least as much as they get in the original coalition structure; for those in coalition C in r's objection, s can guarantee them a payoff no less than what they get from partnering with r.

We can extend similar definitions for a c.r.p.c. First we define the concept of a *partner* of a set of players:

For $K \subseteq N$ and $K \neq \phi$, a player i is called a *partner* of K in a p.c. $(\mathbf{x};\mathcal{B})$, if he is a member of a coalition in $\mathcal{B}$ that intersects K. The set of all partners of K in $(\mathbf{x};\mathcal{B})$ is

$$P[K;(\mathbf{x};\mathcal{B})] = \{i \mid i \in B_j, B_j \cap K \neq \phi\}.$$

Definition Let $(\mathbf{x};\mathcal{B})$ be a c.r.p.c. in a game, and K and L be two disjoint subsets of a coalition $B_j \subset \mathcal{B}$. An *objection of K against L* is a c.r.p.c.
$(\mathbf{y};\mathcal{C}) = (y_1, y_2, \cdots, y_n; C_1, C_2, \cdots, C_l)$
for which
$y_i > x_i$, for all $i \in K$,
$y_i \geq x_i$ for all $i \in P[K;(\mathbf{y};\mathcal{C})]$,
$P[K;(\mathbf{y};\mathcal{C})] \cap L = \phi$.

Definition Let $(\mathbf{x};\mathcal{B})$ be a c.r.p.c. in a game, and $(\mathbf{y};\mathcal{C})$ an objection of K against L, where $K, L \subset B_j$. A *counter objection of L against K* is a c.r.p.c.
$(\mathbf{z};\mathcal{D}) = (z_1, z_2, \cdots, z_n; D_1, D_2, \cdots, D_t)$
for which
$z_i \geq x_i$, for all $i \in P[L;(\mathbf{z};\mathcal{D})]$,
$z_i \geq y_i$, for all $i \in P[L;(\mathbf{z};\mathcal{D})] \cap P[K;(\mathbf{y};\mathcal{C})]$,
$P[L;(\mathbf{z};\mathcal{D})] \not\supset K$.

Notice that in the individual rational case, objections and counter objections are defined as payoff vectors, whereas in the coalitional rational case, they are defined as payoff configurations in order to capture the formation of coalitions.

An objection in an i.r.p.c. is *justified* if it cannot be countered. In $(\mathbf{x};\mathcal{B})$, if player i has a justified objection against player j, we denote this by $i \succ j$. If

i has no justified objection against j, we denote it by $i \not\prec j$. If $i \prec j$, $j \prec k$, and $k \prec i$, then we say that " $\prec$ " is *cyclic* in this case; otherwise, it is called *acyclic*.

Definition An i.r.p.c. $(\mathbf{x}; \mathcal{B})$ is called *i-stable* if each objection of any member v against another member u can be met by a counter objection of u against v. The set of all stable i.r.p.c.'s is called the *bargaining set* M_i of the game.

Similarly, replacing the individual rationality with coalitional rationality, we have:

Definition A c.r.p.c. $(\mathbf{x}; \mathcal{B})$ is called *c-stable* if each objection of a subset K against a disjoint subset L in $(\mathbf{x}; \mathcal{B})$ can be met by a counter objection of L against K. The *bargaining set* M_c of the game is the set of all stable c.r.p.c.'s.

We make the assumption that the players can bargain with perfect information. The logic behind an objection strategy is not that some member in a coalition would necessarily implement it; rather, because of the existence of a counter objection as a credible counter threat, the objections never get implemented. Thus, the stable p.c.'s form a set in which all players have some potential bargaining power to form coalitions to their benefit. The actual outcome would depend on the comparative bargaining skills and power positions of the players.

Now we move from the coalition payoffs to individual payoffs, which is characterized by the notion of an *imputation*:

Definition Suppose N is the set of all n players, and v is the characteristic function of the game. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be an n-tuple of real numbers, where x_i denotes the final payment to player i. Then $\mathbf{x}$ is called an *imputation* of the game, if it satisfies the following:

1. $v(i) \leq x_i$, for every i in N.
2. $\sum_{i \in N} x_i = v(N)$.

Condition 1 stipulates individual rationality, and condition 2 requires group rationality, i.e. the sum of the payoffs to all players should be equal to the the most that the players can get by forming a grand coalition N. If, in addition, we also apply the group rationality to all subsets of N, we obtain the *core* of the game:

Definition An imputation $\mathbf{x}$ is in the *core*, *Co*, of a game if and only if

$$\sum_{i \in N} x_i = v(N) \quad \text{and} \quad v(S) \leq \sum_{i \in S} x_i \ (\forall S \subseteq N).$$

While the core is an appealing solution concept, it can be empty in many games. Moreover, in some large games, the core may be very instable (i.e. change dramatically due to a small perturbation in the payoff structure). Thus we define ε-*core* to mitigate the instability of a core:

Definition For any positive number ε, an imputation $\mathbf{x}$ is in the ε-*core* of a game with characteristic function v, if and only if

$$\sum_{i \in N} x_i = v(N) \quad \text{and} \quad v(S) - \varepsilon|S| \leq \sum_{i \in S} x_i \ (\forall S \subseteq N).$$

Verbally, if $\mathbf{x}$ is in the ε-core, then no coalition S can guarantee each of its members a payoff more than ε above what they get from $\mathbf{x}$.

Next, we define the *excess* of a coalition, a notion that compares the coalition payoff with payoffs for players in that coalition.

Definition The *excess* of a coalition S with respect to an imputation $\mathbf{x}$ is

$$e(S) = v(S) - x(S) \text{ where } x(S) = \textstyle\sum_{i \in S} x_i$$

Thus the excess captures the attitude of coalition S towards the payoff vector $\mathbf{x}$: the bigger the excess, the more deviation the current payoff scheme is to its optimized level, which means the stronger objection there is to the imputation $\mathbf{x}$.

Definition Suppose k and l are two players in the same coalition. The *surplus of k against l at imputation* $\mathbf{x}$ is

$$S_{k,l}(x) = \max_{k \in S,\ l \notin S} e(S).$$

So $S_{k,l}$ represents the biggest gain that k can possibly get from departing $\mathbf{x}$ and form a new coalition without l, given that all other members in that new coalition are satisfied with their payments in $\mathbf{x}$. If $S_{k,l}(x) > S_{l,k}(x)$, we say that k *outweighs* l.

Now we finally get to the definition of a *kernel*, first introduced by Maschler and Peleg [15]:

Definition In a game with coalition structure $\mathcal{B} = B_1, B_2, \cdots B_m$, let $X(\mathcal{B})$ be the set of permissible imputations. The *kernel* of the game is the set of vectors such that for any pair of players (i, j), i does not outweigh j:

$$K(\mathcal{B}) = \{\mathbf{x} \in X(\mathcal{B}) : S_{k,l}(\mathbf{x}) = S_{l,k}(\mathbf{x}) \quad l, k \in B_i \in \mathcal{B}\}$$

Finally, we introduce the *nucleolus* of a game (Schmeidler [28]), which neatly ties together the above concepts.

In an n-person game, we know that there are 2^n possible coalitions, thus 2^n excesses of coalitions with respect to $\mathbf{x}$. We arrange them in descending order, and let $\theta(\mathbf{x})$ be a vector in $\mathbb{R}^{2^n}$ such that

$$\theta(\mathbf{x}) = (e(S_1), e(S_2), \cdots, e(S_{2^n})) \text{ and}$$
$$e(S_1) \geq e(S_2) \geq \cdots \geq e(S_{2^n})$$

Let $\theta_i(\mathbf{x})$ denote the ith component of $\theta(\mathbf{x})$. We say that $\theta(\mathbf{x})$ is *lexicographically* smaller than $\theta(\mathbf{y})$, denoted by $\theta(\mathbf{x}) \ll \theta(\mathbf{y})$, if there exists a positive integer q, such that

$$\begin{aligned} \theta_i(\mathbf{x}) &= \theta_i(\mathbf{y}) \qquad \text{if} \qquad i < q \\ \theta_q(\mathbf{x}) &< \theta_q(\mathbf{y}) \end{aligned}$$

Keep in mind that a smaller excess is more "desirable," thus the decision rule between two outcomes is determined as $\mathbf{x}$ is more acceptable to $\mathbf{y}$, if and only if $\theta(\mathbf{x}) \ll \theta(\mathbf{y})$, and $\theta(\mathbf{x}) \ll= \theta(\mathbf{y})$ means that $\mathbf{x}$ is at least as acceptable as $\mathbf{y}$.

Definition Let X be the set of all imputations in the game with characteristic function v. The *nucleolus* $N(X)$, is

$$N(X) = \{\mathbf{x} \in X : \theta(\mathbf{x}) \ll= \theta(\mathbf{y}), \text{ for all } \mathbf{y} \in X\}$$

Now that we have introduced the notation, we go on to examine some important existence theorems, as well as the inter-relationship among these solution concepts.

5.2.2 Existence Theorem for the Bargaining Set M_i

The notion of M_i is a very important one. It is a bargaining set under the condition that a payoff configuration is individually rational, which as we will see, is a lot weaker than being coalitionally rational. Individual rationality is in line with the assumption of a "rational" person, a basis of the profit-maximization mechanisms of capitalism in classical economics theory.

The Existence Theorem that we will prove in this section is a much stronger statement than merely saying that M_i is non-empty. We will show that for any individually rational coalition structure, there always exists a payoff configuration in the bargaining set M_i.

The existence theorem is ingeniously proved by Peleg [24], who based his proof on a number of lemmas and some results from Davis and Maschler [7]. Some of his arguments are rather brief, so we first fill in the gaps and prove lemmas 5.2.1, 5.2.3 and 5.2.4. For the arguments which Peleg gave a clear explanation, we simply quote his results. The existence theorem will be formally stated at the end of the section.

Lemma 5.2.1 *If in $(\mathbf{x};\mathcal{B})$, player r has a justified objection $\hat{y}^C$ against player s, then each coalition D, for which $s \in D$ and $e(D) \geq e(C)$, must also contain player r.*

Proof We prove the claim by contradiction. Assume in $(\mathbf{x};\mathcal{B})$, player r has an objection $\hat{y}^C$ against player s. Suppose there exists a coalition D with $s \in D$, $r \notin D$ and $e(D) \geq e(C)$, then we want to show that $\hat{y}^C$ is not justified.

First, we construct a payoff vector $\hat{z}^D$ as follows:

$$\begin{cases} z_i = y_i, & i \in C \cap D \\ z_i = x_i, & i \in D - C - \{s\} \\ z_s = v(D) - \displaystyle\sum_{i \in C \cap D} y_i - \sum_{i \in D - C - \{s\}} x_i, & \end{cases}$$

By the setup above, we know that $\displaystyle\sum_{i \in D} z_i = v(D)$. Thus, $\hat{z}^D$ is feasible. Now define δ_s as follows:

$$\delta_s = v(D) - v(C) - \sum_{i \in D - C} x_i + \sum_{i \in C - D} x_i$$

With some algebra, we can show that

$$z_s = x_s + \sum_{i \in C - D} (y_i - x_i) + \delta_s$$

Now, we rewrite $e(D) - e(C)$ as

$$
\begin{aligned}
& e(D) - e(C) \\
= \ & (v(D) - \sum_{i \in D} x_i) - (v(C) - \sum_{i \in C} x_i) \qquad \text{(by definition of excess)} \\
= \ & \sum_{i \in D} (z_i - x_i) - \sum_{i \in C} (y_i - x_i) \\
= \ & \sum_{i \in C \cap D} (z_i - x_i) + \sum_{i \in D - C - \{s\}} (z_i - x_i) + (z_s - x_s) - \sum_{i \in C} (y_i - x_i) \\
= \ & \sum_{i \in C \cap D} (y_i - x_i) + \sum_{i \in D - C - \{s\}} (x_i - x_i) + \sum_{i \in C - D} (y_i - x_i) + \delta_s - \sum_{i \in C} (y_i - x_i) \\
= \ & \delta_s
\end{aligned}
$$

By assumption $e(D) \geq e(C)$, we have $\delta_s \geq 0$. Also, since $\hat{y}^C$ is an objection of r against s, we know $y_i \geq x_i$ (for $i \in C$). Thus, we have

$$z_s = x_s + \sum_{i \in C - D} (y_i - x_i) + \delta_s \geq x_s.$$

We thus showed that $\hat{z}^D$ is a counter objection of s against r's objection $\hat{y}^C$. Therefore, $\hat{y}^C$ is not justified, which contradicts the original assumption. So the claim is proved. ■

Lemma 5.2.2 *Let* $(\mathbf{x}; \mathcal{B})$ *be an i.r.p.c., then the relationship "$\succ$" as defined in the previous section, is acyclic.*

Proof We prove by contradiction. Suppose "$\succ$" is cyclic, then we have for some t players, $1 \succ 2, 2 \succ 3, \cdots, t-1 \succ t, t \succ 1$ $(1, 2, \cdots, t \in B_j \in \mathcal{B})$. Then by definition, we know that there exists a justified objection $\hat{y}^{C_k}$ of player k

against player $k+1$ (mod t), for all $k = 1, 2, \cdots, t$. Since the set of excesses of coalition in $(x; B)$ is finite, we let C_{k_0} be the maximum excess among all the C_k's. Now $\hat{y}^{C_{k_0-1}}$ is a justified objection of $k_0 - 1$ against k_0; by Lemma 5.2.1, we know that C_{k_0} must also contain $k_0 - 1$. By induction, we know that C_{k_0} must then contain all the players, $1, 2, \cdots, t$. However, since $\hat{y}^{C_{k_0}}$ is a justified objection of k_0 against $k_0 + 1$, it cannot contain player $k_0 + 1$ (mod t). Thus the contradiction, and we proved the original claim. ■

Lemma 5.2.3 *The piece-wise maximum (minimum) of a finite number of continuous real-valued functions with a common domain $D \subseteq \mathbb{R}^n$ is continuous.*

Proof We will prove that the maximum of two continuous functions is also continuous. The case with n functions can be easily justified by induction. The minimum case also follows by exactly the same argument.

Suppose $f(x)$ and $g(x)$ are both continuous real functions, with a common domain D.
Let $h(x) = \max(f(x), g(x))$. We want to show that $h(x)$ is also continuous.

Choose $a \in D$ and let x_n be a sequence whose limit is a: $x_n \to a$. By sequential continuity theorem, we know that since both $f(x)$ and $g(x)$ are continuous, given $\epsilon > 0$, there exists $N_1 > 0$, $N_2 > 0$, such that

$$\text{if} \quad n > N_1, \quad \text{then} \quad |f(x_n) - f(a)| < \epsilon;$$

$$\text{if} \quad n > N_2, \quad \text{then} \quad |g(x_n) - g(a)| < \epsilon.$$

We know $h(a) = \max(f(a), g(a))$. There are three cases we need to consider:

1. $f(a) = g(a) = m$, thus $h(a) = m$.

   We know if $n > \max(N_1, N_2)$, then we have $|f(x_n) - m| < \epsilon$ and $|g(x_n) - m| < \epsilon$. Thus,

$$\begin{aligned} m - \epsilon &< f(x_n), g(x_n) < m + \epsilon \\ \Rightarrow m - \epsilon &< \max(f(x_n), g(x_n)) < m + \epsilon \\ \Rightarrow m - \epsilon &< h(x_n) < m + \epsilon \end{aligned}$$

   Thus, $|h(x_n) - h(a)| < \epsilon$ for $n > \max(N_1, N_2)$.

2. $f(a) > g(a)$, thus $h(a) = f(a)$.

   Since $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$, we know there exists $N_3 > 0$ such that if $n > N_3$, then $f(x_n) > g(x_n)$, thus, $h(x_n) = f(x_n)$. Let $n > \max(N_1, N_3)$, we know

$$|h(x_n) - h(a)| = |f(x_n) - f(a)| < \epsilon$$

3. $f(a) < g(a)$, thus $h(a) = g(a)$

   Use the same argument as in the second case.

Hence, we know that for a given ϵ, there exists $N > 0$ such that for $n > N$, $|h(x_n) - h(a)| < \epsilon$. Thus, $h(x)$ is continuous at a. Since a is chosen arbitrarily in D, $h(x)$ is a continuous function on D. Thus, the claim is proved. ∎

Lemma 5.2.4 *Let $\mathcal{B}$ be a c.s. and $i \in B \in \mathcal{B}$. Let $X(\mathcal{B})$ be the set of permissible imputations with c.s. $\mathcal{B}$. Define E_i as*

$$E_i = \{\mathbf{x} \mid \mathbf{x} \in X(\mathcal{B}), \quad i \not\succ j, \quad \forall j \in B\}$$

Then E_i is compact and $E_i \supset \{\mathbf{x} \mid \mathbf{x} \in X(\mathcal{B}), x_i = 0\}$.

Proof We know that each individual payoff x_i is bounded: $v(i) \leq x_i \leq v(N)$, while the end points are attainable. Thus, $X(\mathcal{B})$ is compact. Since E_i is a subset of $X(\mathcal{B})$, it is also bounded. Instead of showing that E_i is closed, we will prove that its complement, $X(\mathcal{B}) - E_i$, is open, which we show by proving that for any $\mathbf{x} \in X(\mathcal{B}) - E_i$, there exists $\varepsilon > 0$, such that if $|\mathbf{x} - \mathbf{z}| < \varepsilon$, then $\mathbf{z} \in X(\mathcal{B}) - E_i$.

We fix $i \in B \in \mathcal{B}$ and let $\mathbf{x} \in X(\mathcal{B}) - E_i$. Choose $j \in B - \{i\}$, j has a justified objection $\hat{y}^Q$ against i in $(\mathbf{x}; \mathcal{B})$. Thus

$$\begin{cases} y_k \geq x_k, & k \in Q \\ y_j > x_j, & \end{cases}$$

Let $y_j - x_j = \epsilon_1 > 0$.

Denote by $\mathcal{D}_{ij}$ the set of all the coalitions that contain player i but not player j: $\mathcal{D}_{ij} = \{D \mid D \subset N, i \in D, \, j \notin D\}$.

Define $f(\mathbf{z})$ for all $\mathbf{z} \in X(\mathcal{B})$ as

$$f(\mathbf{z}) = z_i + \max_{D \in \mathcal{D}_{ij}} (v(D) - \sum_{k \in D \bigcap Q} y_k - \sum_{k \in D - Q} z_k)$$

Thus $f(\mathbf{x})$ is the maximum that i can get when he tries to counter object $\hat{y}^Q$. Since $\hat{y}^Q$ is a justified objection, we know that $f(\mathbf{x}) < x_i$. Let $\epsilon_2 = x_i - f(\mathbf{x}) > 0$.

Since $v(D) - \sum_{k \in D \bigcap Q} y_k - \sum_{k \in D-Q} x_k$ is a linear function of $\mathbf{x}$, it is continuous with respect to $\mathbf{x}$. By Lemma 5.2.3, we know $f(\mathbf{x})$ is continuous. Thus, let $\epsilon_3 = \frac{1}{2}\epsilon_2 > 0$, there exists $\epsilon_3 > \delta > 0$, such that if $\mathbf{z} \in X(\mathcal{B})$ and $|\mathbf{z} - \mathbf{x}| < \delta$, then $|f(\mathbf{z}) - f(\mathbf{x})| < \epsilon_3$.

We know that $|\mathbf{z} - \mathbf{x}| < \delta$ implies $|z_i - x_i| < \delta$. After some simple algebra, we have

$$\begin{aligned} f(\mathbf{z}) - z_i &< f(\mathbf{x}) - x_i + \delta + \epsilon_3 \\ &= -\epsilon_2 + \delta + \epsilon_3 \\ &< -\epsilon_2 + \epsilon_2/2 + \epsilon_2/2 = 0 \end{aligned}$$

Now let $\epsilon = \min(\epsilon_1, \delta)$, we know if $\mathbf{z} \in X(\mathcal{B})$ and $|\mathbf{z} - \mathbf{x}| < \epsilon$, then

$$f(\mathbf{z}) - z_i < 0 \text{ and } y_j > z_j.$$

Thus, $\hat{y}^Q$ is a justified objection of j against i in $(\mathbf{z}; \mathcal{B})$. Therefore, $\mathbf{z} \in X(\mathcal{B}) - E_i$, and we conclude that $X(\mathcal{B}) - E_i$ is open. Thus, the first claim of the proposition is proved.

The second claim of the proposition is fairly straightforward: any player i whose final payoff is 0 always has a counter objection to any objection against him, which he achieves by forming a single-person coalition. Therefore, E_i always contains those whose payoff equals to zero. ∎

Using Lemma 5.2.2 and 5.2.4, the following can be easily shown:

Lemma 5.2.5 *For each* $\mathbf{x} \in X(\mathcal{B})$ *and* $B \in \mathcal{B}$*, there exists* $i \in B$ *such that* $\mathbf{x} \in E_i$.

Based on the above two lemmas, Peleg constructed a set of ingenious functions $c_i(n)$ and used the Brouwer fixed-point theorem (Theorem 4.1.1) to prove the following propositions. Since the proofs were thoroughly presented in [24], we will merely state the results here.

Lemma 5.2.6 *Let* $c_1(\mathbf{x}), \cdots, c_n(\mathbf{x})$ *be* n *non-negative continuous real functions defined on* $X(\mathcal{B})$*. If for each* $\mathbf{x} \in X(\mathcal{B})$*, there is an* $i \in B \in \mathcal{B}$*, such that* $c_i(\mathbf{x}) \geq x_i$*, then there is an* $\hat{\mathbf{x}} \in X(\mathcal{B})$ *such that* $c_j(\hat{\mathbf{x}}) \geq \hat{\mathbf{x}}_j$*, for* $j = 1, 2, \cdots, n$.

Lemma 5.2.7 *Let* $A_1, A_2, \cdots A_n$ *be compact subsets of* $X(\mathcal{B})$*. If* $A_i \supset \{\mathbf{x} \mid \mathbf{x} \in X(\mathcal{B}), x_i = 0, i = 1, 2, \cdots, n\}$*, and for each* $B \in \mathcal{B}, \bigcup_{i \in B} A_i = X(\mathcal{B})$*, then* $\bigcap_{i \in N} A_i \neq \phi$.

Finally, we state the existence theorem for a bargaining set M_i in an n-person game, the proof of which is based on the preceding lemmas.

Theorem 5.2.1 Existence Theorem for Bargaining Set M_i
In an n*-person game with characteristic function* v*, for each coalition structure* $\mathcal{B}$*, there exists an* n*-vector* $\mathbf{x} \in X(\mathcal{B})$*, such that the i.r.p.c.* $(\mathbf{x}; \mathcal{B}) \in M_i$.

Proof As defined in Lemma 5.2.4, $E_1, E_2, \cdots, E_m$ are compact subsets of $X(\mathcal{B})$. By Lemma 5.2.4 and 5.2.5, we know that $E_1, E_2, \cdots, E_m$ satisfy the conditions in Lemma 5.2.7. Hence, $\bigcap\limits_{i \in N} E_i \neq \phi$. Let $\mathbf{x_0} \in \bigcap\limits_{i \in N} E_i$. Therefore, $(\mathbf{x_0}; \mathcal{B})$ is a stable i.r.p.c. that belongs to M_i. ∎

5.2.3 Discussion of the Bargaining Set M_c

The bargaining set M_c, in contrast to M_i, is defined based on coalitional rationality, which is a stricter requirement than individual rationality. Therefore, it naturally follows that $M_c \subset M_i$. It is also quite obvious that M_c is non-empty, since in an n-person game, the p.c. $(0, 0, \cdots, 0; 1, 2, \cdots, n)$ always belongs to M_c. Then it follows that M_i is a non-empty set. Notice, again, Theorem 5.2.1 is a much stronger statement than the non-emptiness of M_i: it states that M_i contains a stable p.c. for *every* individually rational coalition structure.

Aumann and Maschler [4] discussed the bargaining sets M_c (which they call $\mathcal{M}$) for all 2- and 3- person games, as well as some cases in 4-person games. Even in the case of 4-person games, without imposing any restrictions on coalition formation, derivation of the solution in a general form can be very difficult. Thus we will mainly focus on games where only 1-, 2-, and 3-person coalitions are allowed. First, we extend the arguments in [4] and give a full proof of Theorem 5.2.2. Then, we provide an alternative proof of Theorem 5.2.3, an existence theorem of M_c in a 4-person game with $1-$ and $2-$person permissible coalitions.

Lemma 5.2.8 *Assume in an n-person game, $\{12\}$ is a permissible coalition. Let $\mathcal{B} = 12, B_2, B_3, \cdots, B_m$ be a fixed partition of N.*
Let $(\mathbf{x}; \mathcal{B}) = (x_1, x_2, \cdots, x_n; \mathcal{B})$ be a c.r.p.c., and let J be the set of all numbers $\sigma, 0 \le \sigma \le v(12)$, such that player 1 has a justified objection against player 2 in $(\sigma, v(12) - \sigma, x_3, \cdots, x_n; \mathcal{B})$. Then J is an open set relative to the closed

interval[1] $[0, v(12)]$.

Proof We know that $(x_1, x_2, \cdots, x_n; 12, B_2, B_3, \cdots, B_m)$ is a c.r.p.c. For any ε such that $-x_1 \leq \varepsilon \leq v(12) - x_1 = x_2$, let

$$\begin{aligned}(\mathbf{x}'; \mathcal{B}) &= (x_1', x_2', x_3, \cdots, x_n; 12, B_2, B_3, \cdots, B_m) \\ &= (x_1 + \varepsilon, x_2 - \varepsilon, x_3, \cdots, x_n; 12, B_2, B_3, \cdots, B_m),\end{aligned}$$

Since $x_1 + \varepsilon \geq 0$ and $x_2 - \varepsilon \geq 0$, we know $(\mathbf{x}'; \mathcal{B})$ is also a c.r.p.c.

Suppose $x_1 \in J$, and let $\delta = v(12) - x_1$, then we know $\delta > 0$ since otherwise, player 2 can have a counter objection to any objection of 1 by playing alone.

Let $(\mathbf{y}; \mathcal{C}) = (y_1, y_2, \cdots, y_n; \mathcal{C})$ be a justified objection of player 1 against player 2. We know $y_1 > x_1$. Let z_2 be the maximum that player 2 may get by forming a coalition without 1 in a way similar to a counter objection (c.f. $f(\mathbf{z})$ in the proof of Lemma 5.2.4). Since the game is finite, the number of possible coalitions 2 might form is also finite. Thus such a maximum exists and $z_2 < x_2$ (otherwise, player 2 has a counter objection).

Choose ε such that

$$-x_1 \leq 0 < \varepsilon < \min(\delta, y_1 - x_1, x_2 - z_2) \leq v(12) - x_1$$

Thus, we have $y_1 > x_1 + \varepsilon = x_1'$ and $z_2 < x_2 - \varepsilon = x_2'$, then $(\mathbf{y}; \mathcal{C})$ is also a justified objection of player 1 against player 2 in $(\mathbf{x}'; \mathcal{B})$. Therefore, $x_1 + \varepsilon \in J$.

[1] We say a set J is open relative to a closed interval $[a, b]$, if given $x \in J$, there exists $\varepsilon > 0$, such that if $|y - x| < \varepsilon$ and $y \in [a, b]$, then $y \in J$.

Choose ε', such that $0 < \varepsilon' < \varepsilon$. Since $x_1 \in J$, we will show that if $|p - x_1| < \varepsilon'$, and $p \in [0, v(12)]$, then $p \in J$.

We know

$$0 < x_1 - \varepsilon < x_1 - \varepsilon' < p < x_1 + \varepsilon' < x_1 + \varepsilon < v(12)$$

thus,

$$\begin{cases} y_1 > x_1 + \varepsilon > p \\ z_2 < x_2 - \varepsilon = v(12) - x_1 - \varepsilon < v(12) - p \quad (\text{since } \; p < x_1 + \varepsilon) \end{cases}$$

which means that $p \in J$.

Thus we proved that J is an open set relative to $[0, v(12)]$. ∎

Theorem 5.2.2 *In an n-person game, assume all permissible coalitions are either* 1-, 2- *or* 3-*person coalitions, and* $\{12\}$ *is permissible. Let* $\mathcal{B} = 12, B_2, \cdots, B_m$. *If* $(\mathbf{x}; \mathcal{B}) = (x_1, x_2, \cdots, x_n; B)$ *is a c.r.p.c., then there exists a c.r.p.c.*

$$(\mathbf{x}'; \mathcal{B}) = (\sigma_1, \sigma_2, x_3, \cdots, x_n; 12, B_2, \cdots, B_m)$$

such that neither player 1 *nor player* 2 *has any justified objection.*

Proof By Lemma 5.2.8, the payoffs x_1 such that player 1 has a justified objection against player 2 form an open set J_1 relative to $[0, v(12)]$. By symmetry, the payoffs x_1 such that player 2 has a justified objection against player 1 form an open set J_2 relative to $[0, v(12)]$. The claim above is equivalent to saying that there exists a c.r.p.c. in which $x_1 \in [0, v(12)]$ falls outside both J_1 and J_2.

First, we prove that $J_1 \bigcap J_2 = \phi$ by contradiction. Suppose

$$(\mathbf{x}';\mathcal{B}) = (\sigma_1, \sigma_2, x_3, \cdots, x_n; 12, B_2, \cdots, B_m) \in J_1 \bigcap J_2.$$

In other words, $(\mathbf{x}';\mathcal{B})$ is a c.r.p.c. such that player 1 and 2 each has justified objection against the other. Let C_1 be the coalition that 1 forms in a justified objection $(\mathbf{y};\mathcal{C})$ against player 2, and let D_2 be the coalition that 2 forms in a justified objection $(\mathbf{z};\mathcal{D})$ against player 1. Since only 1−, 2−, 3-person coalitions are allowed, we know that C_1 and D_2 each can contain at most three players. Let $C_1 \bigcap D_2 = E$. We know E can contain 0, 1 or 2 players. We examine each possibility:

1. $E = \phi$

   In this case, player 2's objection $(\mathbf{z};\mathcal{D})$ is a valid counter objection against player 1's objection $(\mathbf{y};\mathcal{C})$. Thus, $(\mathbf{y};\mathcal{C})$ is not justified.

2. $E = \{i\}, i \in \{1, 2, \cdots, n\}$

   Let $v_i(E)$ denotes the payoff to players in E in player i's objection ($i = 1, 2$). Without loss of generality, suppose that $v_1(E) \leq v_2(E)$. In this case, we have $v_1(i) \leq v_2(i)$. Again, player 2's objection $(\mathbf{z};\mathcal{D})$ is a valid counter objection. Thus, $(\mathbf{y};\mathcal{C})$ is not justified.

3. $E = \{i, j\}, i, j \in \{1, 2, \cdots, n\}$

   Assume $v_1(E) \leq v_2(E)$, so we have $v_1(i) + v_1(j) \leq v_2(i) + v_2(j)$. There are two possibilities:

   (a)

$$\begin{cases} v_1(i) \leq v_2(i) \\ v_1(j) \leq v_2(j) \end{cases}$$

   In this case, again $(\mathbf{z};\mathcal{D})$ is a counter objection against $(\mathbf{y};\mathcal{C})$.

(b)

$$\begin{cases} v_1(i) > v_2(i) \\ v_1(j) < v_2(j) \end{cases}$$

or

$$\begin{cases} v_1(i) < v_2(i) \\ v_1(j) > v_2(j) \end{cases}$$

By symmetry, we only need to deal with the first set of two inequalities. Let x_2 denote the payoff to player 2 in $(\mathbf{z}; \mathcal{D})$. Now for $D_2 = \{2, i, j\}$,

$$\begin{aligned} v(D_2) &= x_2 + v_2(i) + v_2(j) \quad ((z;\mathcal{D}) \text{ is a c.r.p.c.}) \\ &\geq x_2 + v_1(i) + v_1(j) \quad (\text{by assumption, } v_1(E) \leq v_2(E)) \\ &> \sigma_2 + v_1(i) + v_1(j) \quad ((z;\mathcal{D}) \text{ is an objection by player 2}) \end{aligned}$$

Let $\varepsilon = v(D_2) - v_1(i) - v_1(j) - \sigma_2 > 0$. Construct a payoff vector for players $2, i, j$: $(\sigma_2, v_1(i) + \varepsilon/2, v_1(j) + \varepsilon/2)$. Thus, player 2 can modify $(\mathbf{z}; \mathcal{D})$ to $(\mathbf{z}'; \mathcal{D})$, by changing the payoffs to player $2, i, j$ in $\mathbf{z}$, and it follows that $(\mathbf{z}'; \mathcal{D})$ is a counter objection to $(\mathbf{y}; \mathcal{C})$.

We thus proved that in all three cases, $(\mathbf{y}; \mathcal{C})$ can be countered, which contradicts the assumption that $(\mathbf{y}; \mathcal{C})$ is justified. Thus, $J_1 \bigcap J_2 = \phi$.

Now since both J_1 and J_2 are open sets relative to a closed interval $[0, v(12)]$, J_1 is a union of intervals that are right open, while J_2 is a union of intervals that are left open. Since $v(12) \notin J_1$ and $0 \notin J_2$, there exists $t \in (0, v(12))$, $t \notin J_1$ and $\epsilon_1 > 0$, such that for $0 < \epsilon < \epsilon_1$,

$$t' = t - \epsilon \in J_1.$$

We will show that $t \notin J_2$. Suppose $t \in J_2$. Since J_2 is a union of left open intervals, we know there exists $\epsilon_2 > 0$, such that for $0 < \epsilon < \epsilon_2$,

$$t' = t - \epsilon \in J_2.$$

Therefore, let $\epsilon_3 = \min(\epsilon_1, \epsilon_2)$. For $0 < \epsilon < \epsilon_3$, we have $t' = t - \epsilon \in J_1$ and $t' = t - \epsilon \in J_2$, thus, contradicting $J_1 \bigcap J_2 = \phi$ that we just proved. Hence, we showed that $t \notin J_1$ and $t \notin J_2$, so $(t, v(12) - t, \cdots, x_n; 12, B_2, \cdots, B_m)$ is the c.r.p.c. that we are looking for. Thereby, the original claim is proved. ■

In their paper, Aumann and Maschler went on to give an existence proof in a 4-person game with only 1- or 2-person coalitions permissible, by elaborating on a number of complicated cases. We noticed, however, that a rather short argument based on Theorem 5.2.2 would indeed suffice.

Theorem 5.2.3 *For a* 4*-person game, in which only* 1*- and* 2*-person coalitions are permissible,*

$$\begin{cases} v(1) = v(2) = v(3) = v(4) = 0, \\ v(12) = a, v(23) = b, v(34) = c, \\ v(13) = d, v(24) = e, v(14) = f, \quad a, b, c, d, e, f \geq 0, \end{cases}$$

there always exists a c.r.p.c. $(x_1, x_2, x_3, x_4; 12, 34) \in M_c$.

Proof For a p.c. $(x_1, x_2, x_3, x_4; 12, 34)$, we first show that it is a c.r.p.c. Let $B_1 = \{12\}, B_2 = \{34\}$. There are only two nontrivial subsets of B_1, $B = \{1\}$ or $B = \{2\}$. Thus, by definition of c.r.p.c., if $x_1 \geq v(1) = 0$, and $x_2 \geq v(2) = 0$, then B_1 satisfies the coalitional rationality requirement. Similarly, we know that B_2 is also coalitionally rational. Therefore, $(x_1, x_2, x_3, x_4; 12, 34)$ where $x_i \geq 0$, is a c.r.p.c.

First we fix x_3, x_4. By Theorem 5.2.2, we know that there exists a c.r.p.c. $(\xi_1, \xi_2, x_3, x_4; 12, B_2)$ such that $B_1 = \{12\}$ is stable.

Now we fix ξ_1, ξ_2, and apply Theorem 5.2.2 again to $(\xi_1, \xi_2, x_3, x_4; 12, 34)$. By exactly the same argument, $B_2 = \{34\}$ is stable.

Therefore, there exists $(\xi_1, \xi_2, \xi_3, \xi_4; 12, 34) \in M_c$. ■

In fact, we can generalize the above statement to n-person games partitioned into couples. By induction, it is easy to show that there always exists an imputation $\mathbf{x}$ such that $(\mathbf{x}; \mathcal{B})$ appear in M_c.

There are some interesting similarities and differences between the bargaining set M_c and other solutions, such as ψ-stability and Von Neumann and Morganstern's solution. We will discuss some of the inter-relationships in section 5.4.

5.2.4 Existence and Uniqueness of the Nucleolus

Maschler and Peleg [15] proved the existence of the kernel which is a subset of the bargaining set M_i. Schmeidler [28] showed that the nucleolus is in fact the intersection of the kernel and the non-empty ε-core of a game. Thus, the nucleolus is the third stage in the development of the bargaining set theory of Aumann and Maschler. We will prove the existence and uniqueness of the nucleolus here.

Theorem 5.2.4 Existence Theorem of Nucleolus *Every nonempty compact subset of* $\mathbb{R}^n$ *has a nonempty nucleolus.*

Proof Recall that $\theta^i(x)$ is the i-th largest excess of S with respect to imputation $\mathbf{x}$, where S is a subset of N. By definition, $e(S) = v(S) - x(S)$. Since $v(S)$ is determined independent of $\mathbf{x}$, and $x(S)$ is a linear (thus continuous) function of $\mathbf{x}$, so $e(S)$ is also a continuous function of $\mathbf{x}$. By Lemma 5.2.3, $\theta^i(\mathbf{x})$ is a continuous function of $\mathbf{x}$, for $i = 1, 2, \cdots, 2^n$.

Let Y be a nonempty compact subset of $\mathbb{R}^n$. We define

$$\begin{aligned} Y_1 &= \{\mathbf{x} \in Y \mid \theta^1(\mathbf{x}) \leq \theta^1(\mathbf{y}), \quad \forall\, \mathbf{y} \in Y\} \\ Y_i &= \{\mathbf{x} \in Y_{i-1} \mid \theta^i(\mathbf{x}) \leq \theta^i(\mathbf{y}), \quad \forall\, \mathbf{y} \in Y_{i-1}\} \quad i = 2, 3, \cdots, 2^n. \end{aligned}$$

Since Y is a nonempty compact set, it follows from the Sequential compactness theorem[2] that Y is also sequentially compact. Thus every sequence in Y has a subsequence converging to a point in Y. We first show that Y_1 is also nonempty and compact.

Since $\theta^1 : Y \to \mathbb{R}$ is a continuous function, Y_1 is nonempty, because a continuous function on a nonempty compact set has a minimum.

To show Y_1 is compact, let $\{\mathbf{x_n}\}$ be a sequence in Y_1. Since $\{\mathbf{x_n}\} \subseteq Y_1 \subset Y$, there exists a subsequence $\{\mathbf{x_{n_k}}\}$ that converges to $\mathbf{y_0} \in Y$. We want to show $\mathbf{y_0} \in Y_1$.

We know θ^1 is a continuous function and $\{\mathbf{x_{n_k}}\} \to \mathbf{y_0}$, thus

$$\{\theta^1(\mathbf{x_{n_k}})\} \to \theta^1(\mathbf{y_0})$$

Choose $\mathbf{y} \subset Y$. Since $\{\mathbf{x_n}\} \in Y_1$, $\theta^1(\mathbf{x_n}) \leq \theta^1(\mathbf{y})$ for all n. Because $\{\mathbf{x_{n_k}}\}$ is a subsequence of $\{\mathbf{x_n}\}$, we know $\theta^1(\mathbf{x_{n_k}}) \leq \theta^1(\mathbf{y})$. By limit location theorem,

[2]Sequential compactness theorem states that a compact set is sequentially compact.

$$\theta^1(\mathbf{y_0}) \leq \theta^1(\mathbf{y})$$

Since $\mathbf{y}$ is an arbitrary vector in Y, we conclude $\mathbf{y_0} \in Y_1$. So Y_1 is a nonempty and compact subset of Y. By induction, it is easy to show that Y_i is also nonempty and compact for $i = 2, 3, \cdots, 2^n$.

It is obvious that $Y_{2^n} = N(Y)$. Therefore, we proved that a nonempty compact set has a nonempty nucleolus. ■

Theorem 5.2.5 Uniqueness Theorem of Nucleolus *The nucleolus of a nonempty compact set contains a unique vector.*

Lemma 5.2.9 *Define* $\eta : \mathbb{R}^r \to \mathbb{R}^r$ *by* $\eta(\mathbf{z}) = (\eta_1(\mathbf{z}), \eta_2(\mathbf{z}), \cdots)$ *where*

$$\begin{aligned}
\eta_1(\mathbf{z}) &= \max\{z_t\} = z_{j_1} \\
\eta_2(\mathbf{z}) &= \max\{z_t \mid t \neq j_1\} = z_{j_2} \\
&\vdots \\
\eta_{i+1}(\mathbf{z}) &= \max\{z_t \mid t \neq j_1, j_2, \cdots, j_i\}.
\end{aligned}$$

Then

$$\eta(\mathbf{x} + \mathbf{y}) \ll= \eta(\mathbf{x}) + \eta(\mathbf{y})$$

Proof By definition, η is a lexicographic order of the components of $\mathbf{z}$. Let $\eta_t(\mathbf{x}) = x_{i_t}$, $\eta_t(\mathbf{y}) = y_{j_t}$, $\eta_t(\mathbf{x} + \mathbf{y}) = x_{k_t} + y_{k_t}$ for $t = 1, 2, \cdots, r$.
Obviously, when $t = 1$, then $x_{k_1} \leq x_{i_1}$, and $y_{k_1} \leq y_{j_1}$. Thus we have

$$x_{k_1} + y_{k_1} \leq x_{i_1} + y_{j_1}$$

From the equation above, we have two cases:

1. if $x_{k_1}+y_{k_1} < x_{i_1}+y_{j_1}$, then by definition, we have $\eta(\mathbf{x}+\mathbf{y}) \ll \eta(\mathbf{x})+\eta(\mathbf{y})$.

2. if $x_{k_1} + y_{k_1} = x_{i_1} + y_{j_1}$, then we must have $x_{k_1} = x_{i_1}$, and $y_{k_1} = y_{j_1}$. We move on and look at the second component. Following the same reasoning as in the first case, we know $x_{k_2}+y_{k_2} \leq x_{i_2}+y_{j_2}$. By induction, we know that if and only if

$$\begin{cases} x_{i_t} = x_{k_t} \\ y_{j_t} = y_{k_t} & t = 1, 2, \cdots, r \end{cases}$$

we have

$$\eta(\mathbf{x}+\mathbf{y}) = \eta(\mathbf{x}) + \eta(\mathbf{y}). \tag{5.5}$$

Otherwise, we always have

$$\eta(\mathbf{x}+\mathbf{y}) \ll \eta(\mathbf{x}) + \eta(\mathbf{y}).$$

Thus,

$$\eta(\mathbf{x}+\mathbf{y}) \ll= \eta(\mathbf{x}) + \eta(\mathbf{y})$$

■

Based on Lemma 5.2.9, we prove the uniqueness theorem.

Proof We will use proof by contradiction, and show that if $\mathbf{x}, \mathbf{y} \in N(X)$, i.e. $\theta(\mathbf{x}) = \theta(\mathbf{y})$ and $\mathbf{x} \neq \mathbf{y}$, then

$$\begin{aligned} \theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) &\ll \theta(\mathbf{x}) \quad \text{or} \\ \theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) &\ll \theta(\mathbf{y}). \end{aligned}$$

Which contradicts that both $\mathbf{x}$ and $\mathbf{y}$ are in the nucleolus.

Let $\{v(S)-x(S)\}_{S\subset N}$ be a vector, whose components are $v(S_1)-x(S_1), v(S_2)-x(S_2), \cdots, v(S_{2^n})-x(S_{2^n})$ where $S_1, S_2, \cdots, S_{2^n}$ are all subsets of N. Similarly, $\{v(S)-y(S)\}_{S\subset N}$, and $\{2v(S)-x(S)-y(S)\}_{S\subset N}$ are also vectors in $\mathbb{R}^{2^n}$. We observe the following relationship between θ and η:

$$\begin{aligned}
\theta(\mathbf{x}) &= \eta(\{v(S)-x(S)\}_{S\subset N}) \\
\theta(\mathbf{y}) &= \eta(\{v(S)-y(S)\}_{S\subset N}) \\
2\theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) &= \theta(\mathbf{x}+\mathbf{y}) = \eta(\{2v(S)-x(S)-y(S)\}_{S\subset N})
\end{aligned}$$

Be Lemma 5.2.9, we know that

$$2\theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) \ll= \theta(\mathbf{x})+\theta(\mathbf{y})$$

Thus, either we have

$$2\theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) \ll \theta(\mathbf{x})+\theta(\mathbf{y}),$$

which directly leads to the claimed contradiction, or we have

$$2\theta\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) = \theta(\mathbf{x})+\theta(\mathbf{y}).$$

In the second case, for $t = 1, 2, \cdots 2^n$

$$\begin{aligned}
2\theta^t\left(\frac{1}{2}(\mathbf{x}+\mathbf{y})\right) &= v(S_{k_t})-x(S_{k_t})+v(S_{k_t})-y(S_{k_t}) && (5.6)\\
&= \theta^t(\mathbf{x})+\theta^t(\mathbf{y}) && (5.7)\\
&= v(S_{i_t})-x(S_{i_t})+v(S_{j_t})-y(S_{j_t}) && (5.8)
\end{aligned}$$

By equation (5.5), we know

$$\begin{cases} v(S_{k_t}) - x(S_{k_t}) = v(S_{i_t}) - x(S_{i_t}) \\ v(S_{k_t}) - y(S_{k_t}) = v(S_{j_t}) - y(S_{j_t}) \end{cases}$$

By assumption $\theta(\mathbf{x}) = \theta(\mathbf{y})$, we also have

$$v(S_{i_t}) - x(S_{i_t}) = v(S_{j_t}) - y(S_{j_t})$$

Thus, from equation (5.6) and (5.8),

$$v(S_{k_t}) - x(S_{k_t}) = v(S_{k_t}) - y(S_{k_t}) \quad \text{for } t = 1, 2, \ldots, 2^n$$

which means $x(S) = y(S)$ for all $S \subseteq N$. Therefore, $\mathbf{x} = \mathbf{y}$ which contradicts our original assumption. Therefore, we proved by contradiction the uniqueness of the nucleolus in a nonempty compact set. ■

5.2.5 Examples

In this section, we give some examples to illustrate and review the various solution concepts introduced so far.

Example 5.2.1 *Assume a 3-person game with characteristic function v. The payoff to all coalitions are as follows:*

$$\begin{cases} v(123) = v(12) = 3 \\ v(13) = v(23) = 0 \\ v(1) = v(2) = v(3) = 0 \end{cases}$$

In this game, the coalitional rationality doesn't put any restrictions on the p.c.'s, so $M_i = M_c =$

$$\{(0,0,0;1,2,3), (a, 3-a, 0; 12, 3), (0,0,0;13,2), (0,0,0;1,23), (a, 3-a, 0; 123)\}$$

where $0 \leq a \leq 3$.

$Co = \{(a, 3-a, 0)\}$ $(0 \leq a \leq 3)$.

$K(1,2,3) = \{0,0,0\}$ and $K(12,3) = \{3/2, 3/2, 0\}$. We can verify, for example, the second element:

$S_{1,2} = \max(v(1) - x(1), v(13) - x(13)) = \max(-3/2, -3/2) = -3/2 = S_{2,1}$.

Finally, $N(X) = (3/2, 3/2, 0)$. The nucleolus is quite intuitive. From the game setup, we can see that player 1 and 2 are symmetric, while player 3 has no bargaining power. Thus, the solution reflects such power status, and also conforms to the fairness principle.

Next, we give an example to demonstrate the difference between an individually rational bargaining set M_i, and a coalitionally rational bargaining set M_c.

Example 5.2.2 *Suppose in a 4-person game with characteristic function v, all non-trivial coalitions are as follows:*

$$v(12) = 10, v(13) = 19, v(34) = 20, v(234) = 25$$

Show that $(2,8,0,0;12,3,4)$ is a p.c. in M_i, but not in M_c.

Proof We first show that $(\mathbf{x};\mathcal{B}) = (2,8,0,0;12,3,4) \notin M_c$. Consider an objection by player 1 to player 2, $(\mathbf{y};\mathcal{C}) = (2.5, 0, 16.5, 0; 13, 2, 4)$. Suppose player 2 wants to counter object to $(\mathbf{y};\mathcal{C})$ by $(\mathbf{z};\mathcal{D})$. We know by the definition of a counter objection, the following has to be true:

$$\begin{cases} z_2 \geq 8 \\ z_3 \geq 16.5 \\ z_4 \geq 0 \\ z_2 + z_3 + z_4 = v(234) = 25 \end{cases}$$

Thus, we can see that $\max(z_3+z_4) \leq 17$. However, since $\{34\} \subset \{234\}$, by the definition of c.r.p.c. we know $v(34) \leq z_3+z_4$. Thus, we have $20 \leq z_3+z_4 \leq 17$, which is obviously erroneous. This means $(\mathbf{y};\mathcal{C})$ is justified and we conclude $(\mathbf{x};\mathcal{B}) \notin M_c$.

From the preceding argument, we can see that the contradiction results from the coalitional rational condition $\sum_{i \in B} x_i \geq v(B)$, for all $B \subset B_j \in \mathcal{B}$. Thus if we remove that condition, it can be easily shown that $(\mathbf{x};\mathcal{B}) \in M_i$. The following is a brief explanation:

For every objection $(\mathbf{y};\mathcal{C})$ that player 1 has against player 2, we know that $y_1 > 2$, so $0 \leq y_3 < 17$. Thus, player 2 always has a counter objection $(\mathbf{z};\mathcal{C}) = (0, 8, y_3, 17 - y_3; 1, 234)$ against 1. It is easy to check that $(\mathbf{z};\mathcal{C})$ satisfies all the requirement for an i.r.p.c. A similar argument would apply to any objections by player 2. Therefore, $(\mathbf{x};\mathcal{B}) \in M_i$. ∎

An interesting 6-person game was studied by both Von Neumann and Morgenstern (see [36, page 464]) and Aumann and Maschler [4]. By a rather lengthy and intricate argument, Von Neumann and Morgenstern showed that this simple game only has a trivial solution in M_c. The game setup is as follows:

$$\begin{cases} v(12b) = 1, & b = 3,4,5,6 \\ v(1ab) = 1, & a = 3,4; b = 5,6 \\ v(2pq) = 1, & p = 3, q = 4, \text{or} \;\; p = 5, q = 6 \\ v(3456) = 1, & \\ v(B) = 1, & \text{B contains at least one of the above coalitions} \\ v(B) = 0, & \text{otherwise.} \end{cases}$$

If we loosen the restrictions and consider M_i, it can be shown that some other p.c.'s, such as $(1/3, 1/3, 1/3, 0, 0, 0; 123, 4, 5, 6)$, will also become i-stable.

5.3 Other Solution Concepts

Apart from the cooperative solutions we have introduced in the preceding section, there are some other important and useful concepts, namely, ψ-stability, the Von Neumann and Morgenstern solution and the Shapley value. We will give a brief introduction to each of them in this section.

In the above discussion, we presume that a coalition is stable if no justified changes can be proposed. We have not made any restrictions on what kind of changes is possible. In the real world, there are often times many restraints on the permissible changes in coalition structures. For example, lack of communication between players, irrational behavior or psychological factors, can all be potential reasons why certain changes in collaborative agreements could not happen. The notion of "ψ-stability" (see [11, Chapter 10]), therefore, is based on a rule ψ that determines admissible coalition changes in a game.

Definition Assume a game with characteristic function v, an imputation $\mathbf{x}$, and a coalition structure $\mathcal{B}$. The function $\psi : \mathcal{B} \to \mathcal{B}$ determines the possible changes of coalition structure. The p.c. $(\mathbf{x}; \mathcal{B})$ is *ψ-stable* if

(a). $v(S) \leq \sum_{i \in S} x_i$, for every coalition S in $\psi(\mathcal{B})$.

(b). If $x_i = v(i)$, then in the coalition structure $\mathcal{B}$, player i is not in a coalition with any other players, i.e. $\{i\} \in \mathcal{B}$.

In zero-sum games, a way to easily throw out unfeasible outcomes and shrink the size of the game is by eliminating dominated strategies. Von Neumann and Morgenstern (see [36, page 264]) argued that a stable solution set in cooperative games should also have the property of not being "dominated," which they defined as follows:

Definition An imputation $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ *dominates* an imputation $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ with respect to a nonempty coalition S, if

(a). $v(S) \geq \sum_{i \in S} y_i$

(b). $y_i > x_i$ for any $i \in S$.

Thus, the first condition determines that $\mathbf{y}$ is a feasible condition, and the second condition says that $\mathbf{y}$ is preferred to $\mathbf{x}$ by every player in coalition S.

Definition The *Von Neumann and Morgenstern solution* of a game is a set A of imputations such that

(a). if $\mathbf{x} \in A$, $\mathbf{y} \in A$, then neither $\mathbf{x}$ nor $\mathbf{y}$ dominates the other.

(b). if $\mathbf{z} \notin A$, then there exists $\mathbf{x} \in A$, such that $\mathbf{x}$ dominates $\mathbf{z}$.

It should be noticed that the definition of the solution does not preclude the existence of an imputation outside A that dominates one or more imputations in A. Rather, the focus is on the fact that every imputation outside the solution is dominated by one in it. Thus the solution constructs a dynamic stability: any defector will be punished if he seeks a better individual payoff by deviating away from the solution.

In non-cooperative games, we have seen that by applying the Minimax Theorem, there always exists a unique valuation of the game for each player. If we want to apply a similar idea to cooperative games, we may think of using $v(i)$ as a value for player i. However, as we have shown already, the purpose of coalition making and bargaining is indeed to achieve a higher value than $v(i)$ (which, in most part of this paper, we have assigned to be 0). Thus we need a different procedure to obtain a value for each player. Shapley [30] came up with three reasonable requirements, and proved that if all three conditions are met, there exists a unique evaluation function, which is now referred to as *the Shapley value* of the game for a player.

Let $\phi_i(v)$ denote the value for player i in the game with characteristic function v. Shapley's conditions are

1. In an n-person game, let $\pi : N \to N$, such that π is a one-on-one and onto function, so πv is a game permutated from v. Then we have

$$\pi v(\{\pi(i)|i \in S\}) = v(S) \text{ for all } S \subseteq N,$$

   that is, the $\phi_i(v)$ is independent of a permutation of the players.

2. The individual values of the game forms an additive partition of the value of the grand coalition, i.e.

$$\phi_1(v) + \phi_2(v) + \cdots + \phi_n(v) = v(N)$$

3. For two different games with characteristic functions v and w, ϕ is a linear function:

$$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w).$$

If all three conditions are met, then a unique function can be derived:

Definition The *Shapley value* of player i in a game v is:

$$\phi_i(v) = \sum_{S \subseteq N - \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S)) \tag{5.9}$$

Despite the formidable appearance of the function, in essence, $\phi_i(v)$ is the expected marginal contribution of player i when he enters a coalition, which makes intuitive sense — how much you get depends on how much you are able to contribute.

5.4 Comparison and Evaluation of Different Solutions

The reader may get somewhat confused at this point after being bombarded with numerous notations for different solution concepts. However, it should be noted that these solutions are interrelated in many ways, and it is rare that the reader would find in the literature a paper written on one solution concept without at least mentioning some others. Therefore, we think our effort spent introducing and explaining all these solutions is worthwhile. Given the intricacy of these solutions, in this section, we will compare their rationale, assumptions, and show their respective range of applicability.

We will take a look at the notion of ψ-stability first, as it is more of a normative concept, due to its lack of a concrete functional form. We then proceed with the evaluation in two groups, the bargaining set family (the bargaining set, the kernel, the nucleolus) and the non-coalition-structure based notions (the core, the Von Neumann and Morgenstern solution, the Shapley value). We end this section by discussing results from some experiments and empirical studies.

5.4.1 On ψ-stability

A close look at the definition of ψ-stability reminds us of the core of a game. Indeed, the core is a special (stronger) case of the ψ-stability concept — the core requires $v(S) \leq \sum_{i \in S} x_i$ for every subset S of N, whereas a ψ-stable p.c. only requires S as a subset of the permissible changes of the coalition structure. If $\mathbf{x}$ is in the core, then the p.c. $(\mathbf{x}; 1, 2, \cdots, n)$ is ψ-stable for every ψ. Similar to the core, the coalitional bargaining set M_c is also a special case of the ψ-stability requirement, that is, when each coalition in $\psi(\mathcal{B})$ is a subset of a coalition in $\mathcal{B}$.

The problem with ψ-stability lies in the vagueness of the function ψ. In most cases, it cannot take an explicit form. While it might be possible to qualitatively estimate some features of ψ, there is no guarantee that the players may conform to the estimates. Luce and Raiffa [11, page 226] suggested replacing the rules of permissible changes by "probabilistic statements." Facing the current c.s. $\mathcal{B}$, we can assign a certain probability to the likelihood of a change to coalition S by some players, and such probability can incorporate the restrictions on communication and negotiation skills of the players, which the ψ function was initially introduced to address.

5.4.2 On the Bargaining Set Family

The set of solutions, which we refer to as the "bargaining set family," i.e. the bargaining set, kernel and nucleolus, build upon one another nicely. In contrast to the Nash bargaining solution, we do require transferably utility for the bargaining set family, because these solutions are not covariant with respect to utility transformations.

The theory of the bargaining set takes the formation of coalition structures as *a priori* determined. It addresses the question of "how should players divide their payoffs, given a coalition structure $\mathcal{B}$ is formed?" For example, the bargaining set[3] M contains stable p.c.'s for each feasible coalitional structure. As we have seen in Example 5.2.1, many can be trivial and quite unrealistic. However, we are not arguing that one point in M is superior to another — the motivation for defining a bargaining set is to exclude points outside M because they are unstable. Thus, like the other solutions, the bargaining set helps to narrow down the set of payoff configurations that we can accept as solutions. The solution set is further narrowed down by a subset of M, the kernel, which in turn was proved to contain a unique point, the nucleolus [28].

[3] In some part of this section where the difference between M_i and M_c doesn't make any material difference, we refer to both types of bargaining sets as M for notational convenience.

As we have mentioned before, the purpose of having the objections and counter-objections is *not* to actually carry them out. The bargaining crystalizes at a stage when people do actually want to stay in the coalition they have formed, but want to negotiate their respective shares by leveraging the possibility for them to potentially deviate from the current c.s.

Since the individually rational bargaining set M_i always contains the core (since there are no objections at the core imputations, let alone justified objections), which can be empty for many games, one advantage that M_i has over the core is its non-emptiness for any c.s., as proved in Theorem 5.2.1. On the other hand, the definition of a bargaining set is not without its drawbacks. For example, the definition of an i.r.p.c. (c.r.p.c.) stipulates that only players within a coalition can make objections, which leaves the players who start out alone in a passive position. This again goes back to the question of how coalition structures are formed in the first place, which is not addressed by the theory. We will come back to a brief discussion of this question a little later in this section.

To find a bargaining set for any generic game requires an enormous amount of computation. Maschler [12] has proved that "M_i consists of a finite union of compact convex polyhedra," thus can be determined by solving a system of linear inequalities involving "and" and "or." The number of systems one needs to examine, however, grows exponentially with N. To help solve the computational difficulty of a bargaining set, the kernel was introduced, which considerably reduces the number of inequalities needed. A subset of the bargaining set, the kernel was proved to be an important solution concept on its own, with some desirable mathematical properties (Maschler and Peleg [15]).

Compared with the bargaining set and the kernel, the mathematical definition of the nucleolus is quite complicated. Here we give an intuitive meaning of the solution, borrowed from Maschler, Peleg and Shapley [17]. The excess of a coalition $e(S)$ with respect to the imputation $\mathbf{x}$ is considered as a measurement of dissatisfaction. Thus, locating the nucleolus is an attempt to reduce the biggest dissatisfaction as much as possible (and in the case of a tie, proceed to minimize the second largest dissatisfaction, and so on). While the motivation of this lexicographic order is subject to question (why should we minimize the largest dissatisfaction, instead of, for example, minimizing the average dissatisfaction for all coalitions), we find the nucleolus an appealing concept with its mathematical rigor, its interrelationship with the other concepts (intersection of the kernel and the ε-core when ε-core is not empty), and more importantly, its uniqueness, as proved in Theorem 5.2.5.

The rationale behind the nucleolus is in essence a succession of minimization problems. Applying the same idea, Potters and Tijs [26] did an interesting investigation of "the nucleolus of the zero-sum game," and found some analogous properties to its counterpart for cooperative games.

5.4.3 On Non-Coalition-Structure Based Solutions

In comparison with the bargaining set and the kernel, concepts like the core, Von Neumann and Morgenstern solution and Shapley value are only concerned with imputations, but not coalition structures, which is why we refer to them as "non-c.s. based solutions." It may seem a quite inadequate portrait of the solution of a game, since many times, for example in the case of cartel creation, coalition formation precedes the negotiation within specific coalitions about individual payoffs.

The core is a very straightforward and yet powerful solution concept. There are no objections for the imputations in the core. Because of its simplicity and computational superiority, many consider the bargaining set or the nucleolus as a complementary solution when the core is empty, although Maschler [13] has argued and provided examples in which other points in M_i makes more sense than the core. We will see more of the application of the core in the following chapter.

The Von Neumann and Morgenstern solution is centered on the notion of dominance. The reader might have noticed that condition (a) of the definition of dominance contradicts that of the core (and likewise, of a c.r.p.c.). In non-cooperative games, since the interests of the parties involved are strictly opposed to each other, the complement of a coalition S, $N - S$, will try to drive the payoff of S as low as possible. Therefore, the feasibility condition, $\sum_{i \in S} x_i \leq v(S)$ is reasonable. In cooperative games, however, it is hard to justify why coalition S has to limit its payoff to $v(S)$, since there is no guarantee that the rest of the players will be able to form a coalition, or to act competitively even if the coalition $N - S$ is formed. Therefore, we find it hard to justify the feasibility condition in the Von Neumann and Morgenstern solution for cooperative games.

Like the nucleolus, the Shapley value gives a unique solution to the game. On the other hand, the notion of the Shapley value of a player seems to fall into a different arena than the other solutions — it is not so much a scheme to solve the game and find the equilibrium coalition and payoff vector; rather, since the Shapley value is a weighted average of the incremental additions by player i to all the coalitions i is part of, the value is analogous to an expected return associated with a probability vector, which gives the "value" of a player. Just as the expected value of rolling a fair die is 3.5, a number that would never occur, we do not expect that the Shapley value would necessarily fall in any of the solution concepts we discussed so far.

Finally, we mention a class of games that tie together all the important solution concepts we have studied. Introduced by Shapley [31], the *convex games* are games with characteristic function v, such that for all pairs of coalitions S and T

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

For convex games, both the Von Neumann and Morgenstern solution and the bargaining set coincide with the core; the kernel is contained in the core, and coincides with the nucleolus; the Shapley value is the center of gravity of the core (see Shapley [31], Maschler, Peleg and Shapley [16]).

5.4.4 On Coalition Structure

While historically, notions of the core, the Von Neumann and Morgenstern, and the Shapley value are defined without reference to a coalition structure, Aumann and Dreze [3] connected a given coalition structure $\mathcal{B}$ to these solutions, and made them comparable to the bargaining set family. They were able to find a single function that plays a central role in the theorems for all the solutions except the Shapley value. We found this result quite amazing, in that it is rare in the literature to find a game theoretic function that fits a wide class of solutions.

In the same paper, Aumann and Dreze addressed the question of how a coalition structure $\mathcal{B}$ might form, which is important for the understanding of the solutions that take the formation of a c.s. as given *a priori*. We find their explanation illuminating, and will give a brief summary of their reasoning here.

Aumann and Dreze refuted the traditional arguments for the formation of coalition structures concerning, "the difficulties of communication," "legal barriers such as anti-trust laws" and "personal, family, patriotic, geographical or professional relationships" (p. 128). The authors pointed out that if one were to take these factors into consideration when forming a c.s., they should also be taken into account while the players are comparing their opportunities during the bargaining process. Instead, Aumann and Dreze provided a subtle argument on the reasons for coalition formation: some reasons can be difficult to "measure" and "communicate," and are sometimes "consciously excluded" by the players in the bargaining processes (p. 130). For some players, it might be to their advantage to bargain within the framework of a smaller coalition B, rather than the grand coalition N. Aumann and Dreze illustrated their arguments by an explanatory example of a partition of the academic community into different countries.

5.4.5 Experimental and Empirical Results

Laboratory experiments have been conducted to test under what conditions people adopt the underlying assumptions of the various solutions. An earlier experiment conducted by Kalisch, Milnor, Nash and Nering in 1954 was analyzed in detail by Luce and Raiffa [11]. A more recent series of experiments by Rapoport and Kahan was supportive of the bargaining set and its relevant modifications. The reader is referred to Maschler [13] for a comprehensive reference list of controlled experiments conducted.

Because of the complexity in evaluating characteristic functions and modeling real-life conflicts, evaluations of the applicability of these solutions have been fairly difficult, and relatively scarce. A number of empirical studies on coalitions in various parliaments have resulted in some interesting fits with modified bargaining sets and the nucleolus (Scholfield [29], Peleg [25]).

Chapter 6

Application of Game Theory

In this chapter, we look at how the solution concepts that we have studied so far can be applied to economics contexts to solve real world problems. Our literature review here is not intended to be a comprehensive overview of the conditions under which one solution is more applicable than the others; instead, we will concentrate on a few concepts and see the major areas to which they are applicable. As before, we will keep our focus on the applications in cooperative games. For the purpose of comprehensiveness, we also include a brief discussion on applications of non-cooperative game theory. We will then provide a case study of the airline industry to further illustrate the differences among various cooperative solutions.

6.1 Literature Review

The Nash bargaining solution has been extensively used to model wage negotiations between firms and unions. Alexander and Ledermann [1] analyzed the shapes of the solutions in two cases: when the bargaining is over wages alone, and when it is over both wages and employment. They observed that since the existence of Nash bargaining solutions is defined for two or higher dimensions, when negotiations are over wages only, strict monotonicity conditions need to be imposed to ensure the uniqueness of the solution. The method Alexander and Ledermann developed can be applied to investigate the shape of the Nash bargaining solution for any type of revenue and utility function.

The bargaining set, which takes into consideration the repercussions triggered by a potential objection, has been modified according to different specific situations. Mas-Colell [18] proposed a simplification of the bargaining set and defined it in the continuous case. He was able to prove that under certain conditions, the bargaining set and the set of Walrasian allocations[1] coincide. Based on Mas-Colell's paper, Vind [35] modified the definition of the bargaining set and focused on an atomless[2] exchange economy.

One criticism of the bargaining set is why should we stop at the counter ob-

[1] A Walrasian allocation equilibrium is an allocation pair that is both feasible and optimal given the budget constraints.

[2] An atomless economy means an economy without "big" players, so it can be modeled in continuum. Standard assumptions of an atomless exchange economy include completeness, transitivity, continuity and local non-satiability of preference relations and the strict positivity of the initial endowment to each player.

jections, and not consider the justification of counter- counter objections and so on? To do so, we would need a dynamic system, and hope for some convergence of such a process to the bargaining set. Stearns [33] considered each justified objection as a "demand of definite size" and established a "transfer sequence" accordingly. He proved that "every maximal transfer sequence converges to a point" in the bargaining set, and thus provided a dynamic backup for this important solution concept.

Two major areas of application for the bargaining set family are *cost allocation* and *revenue allocation*. Cost allocation games usually deal with the allocation of the cost of shared resource, such as building a bridge, a tunnel or an airport. For these games, the nucleolus seems to be a popular recommendation. While the nucleolus provides a singular solution, it always requires an exponential number of computations. In solving the problem of building an airport runway, Littlechild [10] provided an algorithm that considerably reduced the needed computation. In a more recent paper, Reinhardt [27] further showed that for a large class of congestion cost allocation games, the computational complexities can be entirely bypassed.

Revenue allocation problems typically concern the division of a specific amount of money, such as the worth of a cooperative enterprise after bankruptcy (Maschler [14]). The most prominent solutions to these problems are the Shapley value and the nucleolus. Surprisingly, in three bankruptcy problems prescribed in the 2000 year-old Babylonian Talmud, the nucleoli coincided precisely with the prescribed solutions. Aumann and Maschler [5] proved that the solution was derived under the criteria of coalition consistency[3].

In addition to concentrating on a few solutions, a number of specific markets have been modeled as cooperative games. In particular, studies on the oligopoly market (Kaneko [9]) and the glove-market (Apartsin and Holzman [2]) have given rise to some interesting results.

Finally, we recognize that non-cooperative game theoretic study has made tremendous contributions to economics, in particular to the development of industrial organization. In fact, game theory has almost become the standard language of industrial organization. Topics for which game theory has proven extremely useful include entry deterrence, predation, collusion and price wars. In addition to providing a general framework for the study of industrial organizations, non-cooperative game theory also forces the modelers to specify assumptions, parameters and relevant information sets, so as to establish a more quantitatively precise model. A thorough discussion of the impact of game theory on industrial organization can be found in Bagwell and Wolinsky [6].

6.2 Case Study on the Airliner Market

In this section, we model the airliner market as a three-person cooperative game. After the merger between Boeing and McDonnell Douglas in 1997, the global market of airplane manufacturing, both for commercial and military

[3]See Aumann and Maschler's paper for the definition and a detailed discussion of "consistency."

use, has become a duopoly market between Boeing and Airbus. While in reality many airline companies may lease, instead of own, some of the aircraft they operate in order to cut down fixed costs, in our model, we simplify the situation, and let all airlines form a grand cartel and act as a monopsony[4] of commercial aircraft. Thus, the three players in the game are Airbus (A), Boeing (B), and this hypothetical cartel of commercial airlines (C). We will give some background information on the market, based on which we set up our model and derive the characteristic functions for different permissible coalitions. We will calculate the core, the bargaining set, the kernel, the nucleolus and the Shapley value of this game, and compare the results to see what implications they have in the real-world market.

6.2.1 Industry Background

Founded in 1926 and currently headquartered in Chicago, the Boeing Company is the largest aircraft manufacturer in the world, with an annual revenue over 50 billion USD. In 2005, Boeing had 55 percent of the total market value of aircraft orders globally. Its $B747$, $B777$, and the upcoming $B787$ Dreamliner are among the most popular models for long-range commercial aircraft.

Airbus, formerly known as *Airbus Intustrie*, began as a consortium of European aviation firms, originally the *Aerospatiale* (France), *Deutsche Aerospace* (Germany) and *CASA* (Spain), in order to compete with the American companies. Right now, Airbus trails Boeing in aircraft manufacturing with a 45

[4] A market situation in which the product or service of several sellers is sought by only one buyer, also referred to as the buyer's monopoly.

percent of market value share, and over 30 billion USD in revenue. In the market for long-range commercial aircraft, the $A340$ family and the newly introduced $A380$ have become strong competitors of their counterparts produced by Boeing.

The airline industry in the U.S. has gone through tremendous changes due to government deregulation and the emergence of low-cost airliners. Recently, many of the major airline companies, including the popular carriers such as *United Airlines*, *US Airways*, *Delta Airlines* and *Northwest Airlines*, have all filed for bankruptcy. Faced with increasing competition and soaring oil prices, these bankrupt and other narrowly escaped airline companies stand at a crucial juncture now, as they make strategic investment decisions to improve efficiency and cut down costs, in the meantime trying to cater to growing customer needs for more terminal points and direct flights.

Any duopoly (oligopoly) market poses the potential risk of cartel formation and implicit collusion among the players. In the case of aircraft manufacturing, if Boeing and Airbus collectively decide to increase prices, the airline companies would have no choice but to bear the increase in cost. Historically, the FTC (Federal Trade Commission) has conducted some investigations into possible collusion between the two companies when both increased aircraft prices[5], although nothing has ever been proved or brought to court. On the other hand, if we assume that all the commercial airlines form a cartel and

[5] For instance, in November 1998, when Boeing revealed plans to increase list prices for new orders by 5 percent, and Airbus by 3 percent, the FTC stepped in to investigate whether it was a result of collusion to fix prices.

act as one unit, then this cartel can deter the potential collusion between the two manufactures by bilateral dealings, or play off one manufacturer against the other. Our model will look at these possibilities using the theories for cooperative games we have developed so far.

6.2.2 Model Setup

There are three players in the game: player A (Airbus), player B (Boeing), and player C (commercial airline cartel). Players A and B are the only manufacturers of long-range commercial aircraft. To simplify the model, we assume that the products are homogeneous, so for C, the marginal utility it obtains from one more airplane is identical regardless of whether the purchase is from A or B.

Player A can decide the price it charges for an aircraft, denoted as p_A. Similarly, player B decides p_B. Player C determines how many aircraft to purchase from each manufacturer respectively, denoted as x_A and x_B, where x_A, x_B are non-negative integers. We assume that the total number of aircraft needed by C is capped at M, which we interpret as the number of long-range aircraft needed to meet the maximum demand by its customers. Therefore, we have:

Assumption 1: $0 \leq x_A + x_B \leq M$.

We denote the manufacturing capacity of player i as m_i, for $i = A, B$. Since Boeing has a larger balance sheet and bigger market share than Airbus, we assume that the manufacturing capacity of player B is at least as big as that of player A. We further assume that the maximum demand of C exceeds the capacity constraint of each manufacturer, but can be satisfied by the total industry production capacity. Thus,

Assumption 2: $0 \leq x_A \leq m_A$ and $0 \leq x_B \leq m_B$.

Assumption 3: $m_A \leq m_B \leq M \leq m_A + m_B$.

Let c_A and c_B be the average cost of producing an aircraft for player A and B respectively. We acknowledge economies of scale in production, therefore, we assume player A is more cost efficient than player B. Let t be the present value of the net profit that an airplane brings to the airline, thus t is exogenously determined. Essentially, we think of t as the sum of a stream of discounted net cash flows generated by an airplane from the date of purchase to its retirement. Notice that t is defined as a net value, so operating costs of an airplane including maintenance and repair have already been deducted.

Obviously, from the manufacturers' point of view, an airplane has to be sold at a higher price than its production cost; from a societal point of view, the production cost of an airplane also has be smaller than its utility value. Thus, we have the last two assumptions:

Assumption 4: $0 \leq c_A \leq p_A$ and $0 \leq c_B \leq p_B$.

Assumption 5: $c_B < c_A < t$.

Given these assumptions and setup above, we have the payoff function f_i for each player i:

$$\begin{aligned} f_A &= f(p_A) = (p_A - c_A)x_A \\ f_B &= f(p_B) = (p_B - c_B)x_B \\ f_C &= f(x_A, x_B) = t(x_A + x_B) - p_A x_A - p_B x_B \end{aligned}$$

Finally, we assume that the players are allowed to have pre-play communication and can form coalitions with one another before they make simultaneous decisions on what prices to charge and what quantities to purchase.

Now we are ready to find the characteristic function, and the value of different coalitions. Recall from the previous chapter that the value of a coalition is the maximum that the coalition can guarantee itself, regardless of what the players outside the coalition will do. Therefore, in this 3-person game, based on our assumptions and the payoff functions, we define and compute the characteristic functions as follows:

$$\begin{aligned} v(\phi) &= 0 \\ v(A) &= v(B) = v(C) = 0 \\ v(AC) &= \max_{\substack{0 \le x_A \le m_A, \\ 0 \le x_B \le m_B}} \{t(x_A + x_B) - p_A x_A - p_B x_B + (p_A - c_A)x_A\} \\ &= (t - c_A)m_A \\ v(BC) &= \max_{\substack{0 \le x_A \le m_A, \\ 0 \le x_B \le m_B}} \{t(x_A + x_B) - p_A x_A - p_B x_B + (p_B - c_B)x_B\} \\ &= (t - c_B)m_B \\ v(AB) &= \max_{c_A \le p_A, c_B \le p_B} \{(p_A - c_A)x_A + (p_B - c_B)x_B\} = 0 \end{aligned}$$

$$\begin{aligned} v(ABC) &= \max_{\substack{0 \le x_A \le m_A, \\ 0 \le x_B \le m_B}} \{(p_A - c_A)x_A + (p_B - c_B)x_B + t(x_A + x_B) - p_A x_A - p_B x_B\} \\ &= (t - c_B)m_B + (t - c_A)(M - m_B) \end{aligned}$$

Since we have already calculated the value of the coalitions {AC}, {BC}, {ABC}, we can reduce the messiness of the writing by denoting the following:

$$\begin{aligned} v(AC) &= (t - c_A)m_A = \alpha \\ v(BC) &= (t - c_B)m_B = \beta \\ v(ABC) &= (t - c_B)m_B + (t - c_A)(M - m_B) = \gamma \end{aligned}$$

From assumptions 3 and 5, we observe the order of magnitude of the values of the characteristic functions:

$$0 = v(AB) < v(AC) < v(BC) < v(ABC)$$

Thus we have simplified and modeled the 3-person cooperative game with two duopoly producers and a monopsony buyer of long-range commercial aircraft into the following 0-normalized game where $N = 3$:

$$\begin{aligned} v(A) = v(B) = v(C) &= 0 \\ v(AB) &= 0 \\ v(AC) &= \alpha \\ v(BC) &= \beta \\ v(ABC) &= \gamma \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $0 < \alpha < \beta < \gamma$.

6.2.3 Solutions to the Game

In this section, we will study the core, the bargaining set, the kernel, the nucleolus and the Shapley value of this game, and discuss the intuitive implications of each solution. The derivations of these solutions are messy but fairly easy algebra, so we omit the computations and provide the results directly.

I. **Core** Co

Following the definition of the core, we know that if x_A, x_B, x_C are the payoffs to players A, B and C, then the following conditions have to be satisfied:

$$\begin{cases} x_A + x_B + x_C = \gamma \\ x_A + x_B \geq 0 \\ x_B + x_C \geq \beta \\ x_A + x_C \geq \alpha \end{cases}$$

Recall that the core consists of a set of imputations, which in the current case is $\mathbf{x} = (x_A, x_B, x_C)$. The core Co is not directly related to any specific coalition structure, and in this game it is computed as

$$Co = \begin{cases} Conv\{(0,0,\gamma), (\gamma-\beta,0,\beta), (0,\gamma-\alpha,\alpha), (\alpha,\gamma-\alpha,0), (\gamma-\beta,\beta,0)\} \\ \qquad \text{if} \quad \gamma > \alpha+\beta \\ Conv\{(0,0,\gamma), (\gamma-\beta,0,\beta), (0,\gamma-\alpha,0), (\gamma-\beta,\gamma-\alpha,\alpha+\beta-\gamma)\} \\ \qquad \text{if} \quad \gamma \leq \alpha+\beta \end{cases}$$

A careful observation of the core indicates that player B has a slight advantage over player A (notice in both cases, the maximum payoff that player B can get, $\gamma-\alpha$, is greater than the maximum payoff player A can get, $\gamma-\beta$), which is in accordance with our expectation. Since Boeing has a larger market share and lower cost compared with Airbus, it consequently obtains a better bargaining position than Airbus.

II. **Bargaining set** M_c

The complete formula for a coalitionally rational bargaining set M_c for a 3-person game is derived in [4]. For our game, the bargaining set M_c is computed as:

$$M_c = \begin{cases} (0,0,0;A,B,C) & \\ (0,\beta - x_c, x_c; A, BC) & 0 \leq x_c \leq \beta \\ (0,0,\alpha; AC, B) & \\ (0,0,0;AB,C) & \\ (Co; ABC) & \end{cases}$$

Notice that in the case when all players form a grand coalition, the imputation is the same as the core, as computed above. We observe that except for the last case, all other i.r.p.c.'s give player A a final payoff of 0. Therefore, it is in the interest of Airbus to avoid unilateral dealing with the cartel of the commercial airlines — a somewhat counter-intuitive result — but instead, try to facilitate three-sided talks and form a grand coalition.

III. **Kernel** K

Recall that the kernel is a subset of the set of permissible imputations with coalition structure $\mathcal{B}$. In the kernel, the surpluses of any two members in the same coalition against each other always have to be equal. Since the kernel is a subset of the bargaining set, we check each solution in the bargaining set to determine the kernel.

Also recall from section 5.4.3 that for convex games, the kernel coincides with the nucleolus $N(v)$. In this game, for the grand coalition $\mathcal{B} = \{ABC\}$, when $\gamma \geq \alpha + \beta$, the game turns out to be a convex game. Therefore, the kernel is given by:

$$\begin{cases} K(A,B,C) = (0,0,0) & \\ K(A,BC) = (0,(\beta-\alpha)/2,(\alpha+\beta)/2) & \\ K(ABC) = ((\gamma-\beta)/2,(\gamma-\alpha)/2,(\alpha+\beta)/2) & \text{when} \quad \alpha+\beta > \gamma \\ K(ABC) = N(v) & \text{when} \quad \alpha+\beta \leq \gamma \end{cases}$$

We observe that the kernel K eliminates two stable p.c.'s in the bargaining set, which are constructed based on the coalition structures $\{AC\}$ and $\{AB\}$. This indicates that the commercial airline cartel has such huge monopsony power that it doesn't even pay for the two duopoly producers to form a coalition to fix prices — the enticement from C to B is big enough to render the coalition $\{AB\}$ unstable. When B and C do make a coalition, it is obvious that C has an upper hand, which guarantees it a much better payoff.

IV. **Nucleolus** N

The nucleolus N is the unique intersection of the kernel K and the core Co (which in the current game is nonempty). Based on the algorithm derived in [10], the nucleolus of this game is as follows (cf. Shenoy [32]):

$$N = \begin{cases} (\gamma/3,\gamma/3,\gamma/3) & \text{when} \quad \gamma > 3\beta \\ ((\gamma-\beta)/2,(\gamma+\beta)/4,(\gamma+\beta)/4) & \text{when} \quad \beta+2\alpha \leq \gamma \leq 3\beta \\ ((\gamma-\beta)/2,(\gamma-\beta)/2,(\alpha+\beta)/2) & \text{when} \quad \alpha+\beta \leq \gamma \leq \beta+2\alpha \\ ((\gamma-\beta)/2,(\gamma-\alpha)/2,(\alpha+\beta)/2) & \text{when} \quad \beta < \gamma \leq \alpha+\beta \end{cases}$$

We notice that when γ, the value of coalition $\{ABC\}$, substantially outweighs β, the value of coalition $\{BC\}$ (i.e. $\gamma > 3\beta$), the payoffs to the three players are identical. This is because it is in the mutual interest of all the players to form a grand coalition, so C's bargaining advantage cannot actually be demonstrated. As γ and β get closer, however, the relative payoff to C increases, since C has to be compensated for preferring coalition $\{ABC\}$ to coalition $\{BC\}$. The closer γ is to β, the more obvious is C's bargaining advantage.

V. **Shapley value** $\phi_i(v)$

We compute the Shapley value to each player according to equation (5.9):

$$\begin{aligned}
\phi_A &= (2\gamma + \alpha - 2\beta)/6 \\
\phi_B &= (2\gamma - 2\alpha + \beta)/6 \\
\phi_C &= (2\gamma + \alpha + \beta)/6
\end{aligned}$$

We observe that $\phi_A + \phi_B + \phi_C = \gamma$ and $\phi_A < \phi_B < \phi_C$. The Shapley value also dictates that the commercial airline cartel has an advantage over Boeing, which in turn has an advantage over Airbus. As we have discussed in section 5.4.3, the Shapley value is entirely based on the characteristic function v rather than the bargaining power of the players in the process of coalition formation. Thus, we interpret the Shapley value as a "normative" solution, a value that rational players should accept on the ground of some fairness principle — how much you are entitled to obtain should depend on how much you can contribute to a coalition.

6.2.4 Limitations of the Model

Our model of the long-range aircraft market is undoubtedly highly simplified. The biggest disputable assumption we made is that all commercial airlines form a cartel in the bargaining process. This is obviously impossible to achieve in the real world, although with 4 out of the 7 major U.S. airlines already filed for Chapter 11 bankruptcy and taken over by the government, it is not entirely unrealistic that some form of government re-regulation of the airline industry might happen soon.

Another strong assumption we made is the homogeneity of the products. Although long-range aircraft constitute a rather specific and inelastic segment of the airliner market, we are not familiar with the decision procedure of airliners when they make aircraft purchases. We believe that criteria in addition to price play an indispensable role in the purchase decision on airplane models, but our limited knowledge of the industry prevents us from quantifying the other relevant factors.

Despite the above caveats for our model, this case study gives us a chance to study different solution concepts for cooperative games in a real-world setting. We have obtained a number of useful, and sometimes counter-intuitive, observations, and gained a deeper understanding of the applied value of these solutions. We hope modifications of our model, for example into an n-person cooperative game, will more accurately reflect the real situation.

Chapter 7

Conclusion

With a focus on cooperative games, this paper is by no means intended to be a comprehensive study of game theory, although I do try to include some of the most important theorems (e.g. Minimax Theorem) and well-known solutions (e.g. Nash Equilibrium) for non-cooperative games. Within cooperative games, I studied in detail the Nash bargaining solution and the bargaining set family, again with a focus on the latter. I gave some intuitive explanations of the rationale behind each solution in the bargaining set family, evaluated the assumptions of different cooperative solutions, and compared them qualitatively based on a few numerical examples as well as a mathematical model I constructed in a case study of the airliner industry. To this end, I hope the reader will obtain, by now, an overview of the broad structure of game theory, and a general understanding of the intricate interrelationship and differences among the solution concepts for cooperative games, and their respective range of application.

In response to the basic question I raised in the beginning, "*Under what circumstances would some solutions we have studied be preferred to others?*", I do not want to give the impression that either the bargaining set, or the nucleolus is superior to any other solutions in any particular case. We have seen that each solution sheds some light on one aspect of the real world problem, and by examining all these solutions, we can get a better picture of the issues involved. Admittedly, however, as I have mentioned, the computational difficulty gets in the way as soon as the number of players gets larger. I feel that even though some solutions cannot be fully computed, certain properties may still bring interesting insights on applications. Just as the computation of the nucleolus has been radically simplified over the years (Littlechild [10], Reinhardt [27]), I hope better algorithms will be developed to enable more efficient computation of the other solutions.

It appears to me that non-cooperative game theory has already become a fairly developed tool for economic application, and recent major progress on the mathematical end falls mostly into the area of cooperative games. In addition to the improvement of computational efficiency, current research is focusing on developing theorems for specific types of games and markets, as I briefly described in section 6.1. As Rubinstein pointed out in his afterword for the sixtieth anniversary edition of the landmark work by Von Neumann and Morgenstern [36], "the last decade has seen few new ideas in game theory ... the stage is set for a new unconventional work," I hope that new concepts like that of Nash equilibrium will come up as additional pillars in the arena of cooperative games, and game theory will continue to enjoy remarkable expansion as it did since its inception sixty years ago.

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