Seidel Switching

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Abstract

Seidel switching is a technique for generating pairs of graphs that are cospectral [1]but not necessarily isomorphic. We will discuss and prove some important properties related to this graph construction. Cospectral pairs of regular graphs are rarer than cospectral pairs of non-regular graphs. As a result, after looking at how to construct graphs via the Seidel technique, we will look specifically at generating regular pairs of graphs via the Seidel swtiching technique.

[1]In [7] the term *isospectral* is used, while other sources use *cospectral*. Since we will be referring to pairs of graphs, the term cospectral will be used for consistency.

Contents

1 Introduction

We will be exploring a topic in graph theory known as Seidel switching. It was first introduced in [9] as part of a discussion on equilateral point sets in elliptic space and later explored by Robert Brooks and Gregory Quenell. In [7], Gregory Quenell explored Seidel switching in order to answer the question "can you hear the shape of a graph?" In other words, does the spectrum of a graph determine the structure (or shape) of a graph? In proving that Seidel switching generates cospectral graphs that are not necessarily isomorphic, we can see that the answer to this question is no, graph structure is not determined by its spectrum. This question is directly related to the question posed by Marc Kac in 1966, "can you hear the shape of a drum?" This question explores whether or not a manifold can be determined uniquely by the spectrum. We will be taking the work of Quenell in [7] and expanding on it. Providing proofs of facts and theorems stated, as well as specific examples of facts mentioned throughout.

We will begin by introducing background information to be used throughout. This includes definitions, theorems and proofs that will be important to our results. We denote graphs as G or Γ. These graphs are undirected, may contain more than one component and we allow for loops and multiple edges. We will then describe the Seidel switching construction and work through a proof of the fact that graphs constructed via the Seidel technique are cospectral by providing a nontrivial example illustrating the ideas presented. We will also look at the construction with a restriction on it to generate non-isomorphic pairs of regular graphs that are cospectral. We will discuss a conjecture that arises naturally from certain examples and will consider a function, called the Ihara zeta function, that gives us information about the graphs. Throughout we will discuss various properties that arise from Seidel switching and provide specific examples to illustrate these.

2 Definitions

Before beginning our discussion of Seidel switching, we introduce some definitions that will be helpful.

Definition 2.1. *A* ***graph*** $G = (V, E)$ *is a collection of vertices* $v_i \in V$ *and edges* $\epsilon_i \in E$ *such that each edge has a vertex at each of its endpoints for edges* $\epsilon_i = \{v_j, v_k\}$ *where* $v_j, v_k \in V$*.*

Definition 2.2. *The* ***degree****, or* ***valency****, of vertex* v *in a graph* G *is the number of edges incident to* v*, where each loop at* v *is counted twice.*

Definition 2.3. G *is* ***regular*** *if each vertex is of the same degree, and* q***-regular*** *if the common degree is* q*.*

Definition 2.4. *For a graph* G*, a* ***walk*** *from vertex* v *to vertex* w *is a sequence of edges*

$$\epsilon_1\epsilon_2\epsilon_3 \ldots \epsilon_n$$

such that the initial vertex of ϵ_1 *is* v *and the terminal vertex of* ϵ_n *is* w *where the terminal vertex of* ϵ_j *is the initial vertex of* ϵ_{j+1}*. A* ***closed walk*** *occurs when the initial vertex is the same as the terminal vertex of the walk. We refer to a walk such that* n=k *as a* k***-walk****.*

Definition 2.5. *A* ***prime walk*** C *is a closed walk without backtracking or a* *tail (backtracking that occurs at the last step) and there does not exist a walk B such that $C = B^k$ for $k > 1$[6].*

Definition 2.6. *The* ***adjacency matrix*** *A of G is the symmetric $n \times n$ matrix in which entry $a_{ij} = a_{ji}$ is the number of edges in G with endpoints (v_i, v_j). (Note that the diagonal entry a_{ii} is twice the number of loops at vertex i. This is because for an undirected graph, a loop can be traversed in either of two directions.)*

Definition 2.7. *The set of eigenvalues of A is the* ***graph spectrum*** *of G.*

Definition 2.8. *The* ***length spectrum*** *of G is the sequence*

$$l_0(G), l_1(G), l_2(G)...$$

where $l_k(G)$ is the number of closed k-walks in G for integers $k \geq 0$.

Definition 2.9. ***Isospectral*** *graphs are graphs with the same graph spectrum, or their adjacency matrices have the same eigenvalues.*

Definition 2.10. *If a pair of graphs are isospectral, they are called* ***cospectral*** *graphs [10].*

Definition 2.11. *Graphs are* ***length isospectral*** *if they have the same length spectrum.*

Definition 2.12. *A* ***complete graph*** K_n *is a graph where each pair of vertices is connected by an edge.* K_n *will have* n *vertices,* $\frac{n(n+1)}{2}$ *edges and is regular of degree* $n-1$.

Definition 2.13. *Two graphs* G_1 *and* G_2 *are said to be* ***isomorphic*** *if there is a 1-1 and onto mapping* $f : V_1 \to V_2$ *such that* $\{v_i, v_j\} \in E_1$ *if and only if* $\{f(v_i), f(v_j)\} \in E_2$.

Definition 2.14. *The* ***automorphism group***, ***Aut(G)***, *of a graph* G *is the group of isomorphisms from* G *to* G.

Definition 2.15. *The* ***symmetric group*** S_n *of degree* n *is the group of all permutations on* n *elements.* S_n *is therefore a permutation group of order* $n!$.

We note that every group of order n is isomorphic to a subgroup of S_n.

Definition 2.16. *The* ***dihedral group*** D_4 *is the group generated by the permutations* (1234) *and* (13) *and corresponds to the symmetries of the square.*

Definition 2.17. *A graph* Γ *is* ***md2***, *or minimal degree 2, provided the degree of each vertex is at least 2.*

Hence such a Γ is not a tree and does not contain any vertices of degree zero.

Definition 2.18. *The* ***Ihara zeta function*** *of a regular graph* Γ *is given by* *the following infinite product*

$$\mathbf{Z}_\Gamma(u) = \prod_{primes\ C\in\Gamma} (1 - u^{degC})^{-1}$$

for prime walks C of Γ. Where deg C is the number of edges in C.

We also have the following:

Theorem 2.19. ***Ihara's Theorem:*** *If Γ is a connected $(q+1)$-regular graph with adjacency matrix A, n vertices and e edges, then the Ihara zeta function is defined as the following rational function*

$$Z_\Gamma(u) = \frac{(1-u^2)^{n-e}}{\det(I - Au + Qu^2)}.$$

Here $Q = qI$ is the diagonal matrix where d is the degree of the vertices.[2]

For a proof of Ihara's Theorem see [8] pp.417-418.

[2]We use $Q = qI$ because Γ is regular. For a non-regular graph, Q is the diagonal matrix such that each a_{ii} along the diagonal is equal to the degree $q+1_i$ minus 1 at each vertex v_i of Γ.

3 Eigenvalues and k-walks

The theorems and corresponding proofs of this section are an important part of [7] to show that two graphs have the same length spectrum if and only if they have the same spectrum. This will then be used in the proof that Seidel switching produces cospectral graphs. One conclusion that will we be using in this section is the fact that because A is symmetric, its eigenvalues will be real. For a proof of this, we refer you to section B of the appendix.

Theorem 3.1. *Let $A = (a_{ij})$ be the adjacency matrix of an undirected graph G. For each integer $k \geq 0$, the number of k-walks from v_i to v_j in G is equal to*

$$[A^k]_{ij},$$

the ij^{th} entry of the k^{th} power of A.

Proof. We will prove this theorem using induction.

Let G be an undirected graph with vertices $v_1, v_2, ..., v_n$. Let A be the adjacency matrix of G.

We will show that for all integers $k \geq 0$,

$$[A^k]_{ij} = \text{the number of } k\text{-walks from } v_i \text{ to } v_j$$

for all vertices of G.

$n = 1$:

$$A^1 = A \Rightarrow [A^1]_{ij} = [A]_{ij}$$

The entry a_{ij} equals the number of edges between v_i and v_j (by the definition of adjacency matrix), but this is the same as the number of 1-walks from v_i to v_j for $i \neq j$.

We recall from earlier that for a loop at v_i, we count two 1-walks from v_i to v_i since a loop contributes to the valency twice.

$n = k - 1$: Assume $[A^{k-1}]_{ij}$ =the number of $k - 1$-walks from v_i to v_j. (This is our induction hypothesis.)

$n = k$: Assuming the induction hypothesis, we want to show that

$$[A^k]_{ij} = \text{the number of } k\text{-walks from } v_i \text{ to } v_j.$$

Let $A = (a_{ij})$ and $A^{k-1} = (b_{ij})$. So

$$A^k = AA^{k-1} = A^{k-1}A$$

and the ij entry of A^k can be found by multiplying the i^{th} row of A by the j^{th} column of A^{k-1}. Hence,

$$[A^k]_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$$

for all $i, j \in \{1, 2, ..., n\}$.

We will now look at each term of the above sum:

$$a_{i1} = \text{the number of edges between } v_i \text{ and } v_1$$

$$b_{1j} = \text{the number of } k-1\text{-walks between } v_1 \text{ and } v_j$$

$$\text{(by the induction hypothesis).}$$

Clearly, any edge between v_i and v_1 can connect with any $k-1$-walk from v_1 to v_j and form a k-walk from v_i to v_j with v_1 as its second vertex.

By multiplication, we now have

$$a_{i1}b_{1j} = \text{the number of } k\text{-walks from } v_i \text{ to } v_j \text{ in } G$$

$$\text{with } v_1 \text{ as the second vertex of the walk.}$$

We may now generalize this for all $m \in \{1, 2, ..., n\}$ to obtain the following:

$$a_{im}b_{mj} = \text{the number of } k\text{-walks from } v_i \text{ to } v_j \text{ in } G$$

$$\text{such that } v_m \text{ is the second vertex of the walk.}$$

For every k-walk from v_i to v_j there is a vertex of G that is the second vertex of the walk. Thus, the total number of k-walks from v_i to v_j is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}.$$

From earlier, we know that this is $[A^k]_{ij}$. Thus,

$$[A^k]_{ij} = \text{the number of } k\text{-walks from } v_i \text{ to } v_j \text{ in } G.$$

□

We can now use this conclusion to prove the following theorem that will indicate the direct relationship between the spectrum and length spectrum of a graph G.

Theorem 3.2. *Let G be a graph with adjacency matrix A and let $\lambda_1, \lambda_2, ..., \lambda_N$ be the eigenvalues of A. For each integer $k \geq 0$, the total number of closed k-walks in G is equal to*

$$\lambda_1^k + \lambda_2^k + ... + \lambda_N^k.$$

Proof. The total number of closed k-walks in G equals the sum of the number of closed k-walks at each vertex v_i in G. A closed k-walk at a vertex $v_i = [A^k]_{ii}$.

Let us recognize the following lemma and corresponding proof.

Lemma 3.3. *For a diagonalizable matrix A and its corresponding diagonal matrix $D = QAQ^{-1}$, tr(A)=tr(D).*

Proof. We first show $tr(AB) = tr(BA)$.

$$\sum_i [AB]_{ii} = \sum_i \sum_k a_{ik} b_{ki}$$

$$= \sum_k \sum_i b_{ki} a_{ik}$$
$$= \sum_k [BA]_{kk}$$

We now apply $tr(AB) = tr(BA)$ to get

$$tr(QAQ^{-1}) = tr(Q\ AQ^{-1})$$

$$= tr(AQ^{-1}\ Q)$$

$$= tr(A)$$

$\square$

This tells us, since the entries along the diagonal of a diagonal matrix are its eigenvalues, $tr(D)$=the sum of the eigenvalues= $tr(A)$.

We now have the total number of closed k-walks in $G =$

$$\sum_{i=1}^{N} [A^k]_{ii} = tr(A^k).$$

Since λ_i are eigenvalues of A, λ_i^k are eigenvalues of A^k.

Thus, we have

$$tr(A^k) = \lambda_1^k + \lambda_2^k + ... + \lambda_N^k.$$

$\square$

4 An introduction to Seidel switching

Let us begin by explaining how to construct a pair of cospectral graphs using the Seidel switching technique. This can also be found in [7] pp. 5-6.

Given the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that G_2 is regular and has an even number of vertices, we create a set of edges $\mathcal{E}$. The set $\mathcal{E}$ joins each vertex in V_1 to half of the vertices in V_2. There is no predetermined way to choose $\mathcal{E}$. We also form the set of edges $\mathcal{E}^{\mathcal{C}}$ such that there is an edge in $\mathcal{E}^{\mathcal{C}}$ between a vertex in V_1 and a vertex in V_2 if and only if there is no edge between these vertices in $\mathcal{E}$.

These two edge sets form two different graphs Γ_A and Γ_B such that

$$\Gamma_A = (V_A, E_A) = (V_1 \cup V_2, E_1 \cup E_2 \cup \mathcal{E})$$

and

$$\Gamma_B = (V_B, E_B) = (V_1 \cup V_2, E_1 \cup E_2 \cup \mathcal{E}^{\mathcal{C}})$$

We provide illustrations to show this construction more clearly. We begin with G_1 and G_2 as shown in Figure 4.1 on p.16a. Note that G_2 has an even number of vertices and is 2-regular. We then introduce the sets $\mathcal{E}$ and $\mathcal{E}^{\mathcal{C}}$ in Figure 4.2 on p.16a. We can now construct the graphs Γ_A and Γ_B as shown in Figure 4.3 on p.16a. We see that in this example, Γ_A and Γ_B are non-isomorphic. To see

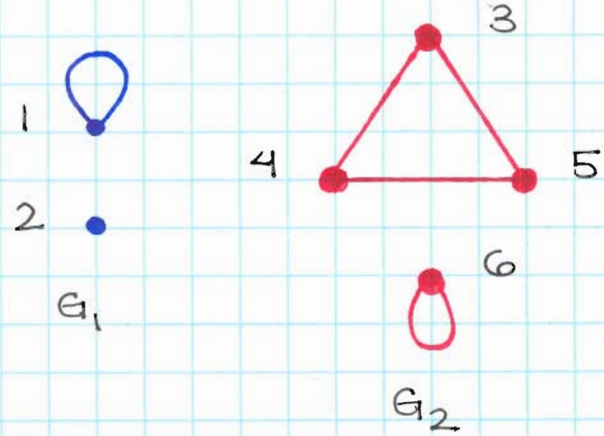

Figure 4.1

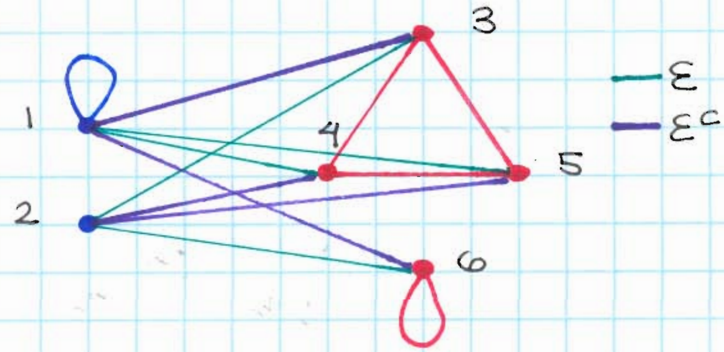

Figure 4.2: G_1 and G_2 with $\mathcal{E}$ and $\mathcal{E}^c$

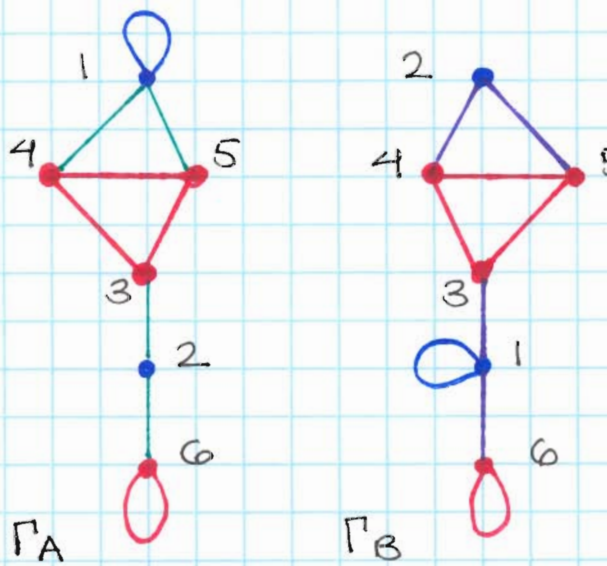

Figure 4.3: Γ_A and Γ_B

this, we can compare the location of the loops in Γ_A and Γ_B.

Definition 4.1. *We can refer to the construction involving* G_1*,* G_2*,* $\mathcal{E}$*,* $\mathcal{E}^{\mathcal{C}}$*,* Γ_A*, and* Γ_B *as the* ***Seidel pair*** $(G_1, G_2, \mathcal{E})$*.*

Definition 4.2. (Γ_A, Γ_B) *is a* ***Seidel switch*** *of* (G_1, G_2)*.*

The dependence on $\mathcal{E}$ is important because a different choice of the edge set $\mathcal{E}$ combined with G_1 and G_2 can create a different pair of Γ_A and Γ_B as the following example shows. While [7] alludes to this fact by noting that $\mathcal{E}$ uniquely identifies a Seidel pair, it does not directly state the following:

Example 4.3. *Given graphs* G_1 *and* G_2*, there may exist more than one choice for* $\mathcal{E}$*.*

Figures 4.4-4.8 on pp.16b-c illustrate this example. We can see that we begin with the same G_1 and G_2 such that they both have four vertices and G_1 and G_2 are both 2-regular. However, we choose two different edge sets $\mathcal{E}$, as shown, that generate two different Seidel switches of (G_1, G_2). In this case we produce two pairs of isomorphic graphs.

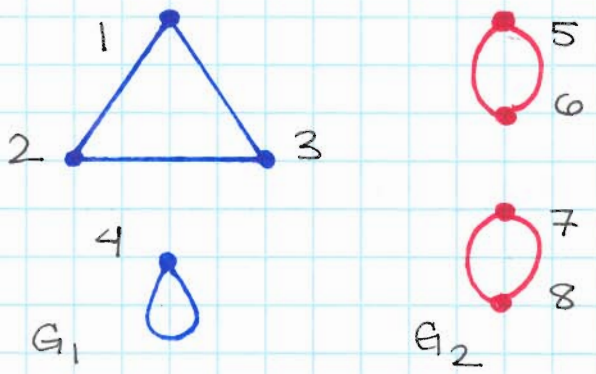

Figure 4.4

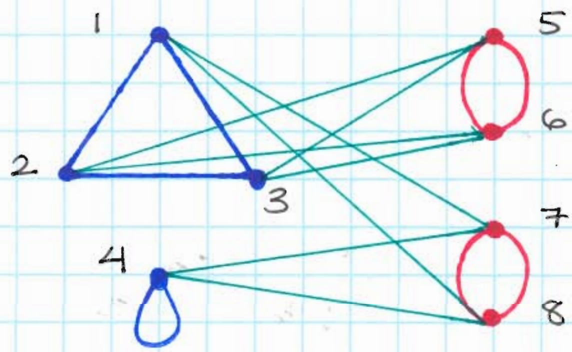

Figure 4.5: G_1 and G_2 with first choice of $\mathcal{E}$

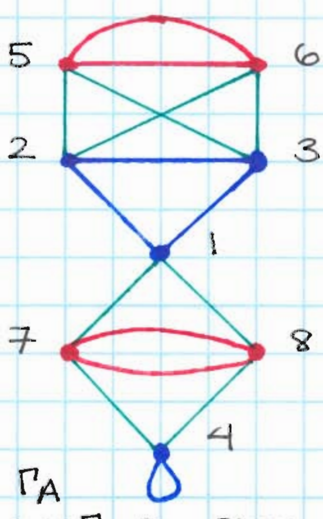

Figure 4.6: Γ_A for first choice of $\mathcal{E}$

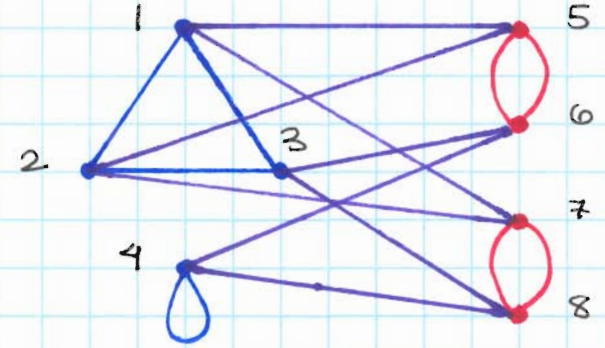

Figure 4.7: G_1 and G_2 with second choice of ε

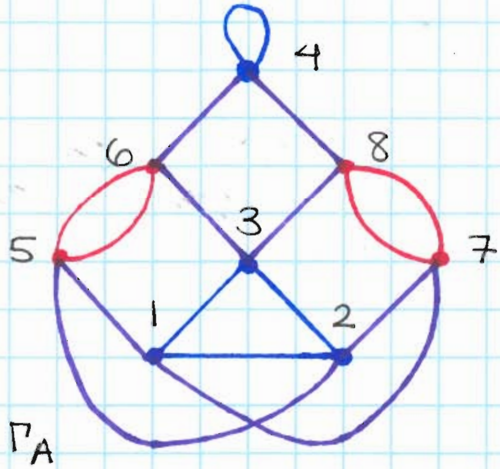

Figure 4.8: Γ_A for second choice of ε

5 Seidel switching generates cospectral graphs

Theorem 5.1. *Graphs constructed via the Seidel technique are cospectral.*

Section 4 of [7] provides a combinatorial proof that shows graphs constructed via the Seidel technique are length isospectral and thus cospectral. We now outline this proof and illustrate it with the following nontrivial example.

We begin with the Seidel pair $(G_1, G_2, \mathcal{E})$ as shown in Figures 5.1-5.3 on pp.18a-b. We then provide (v, w, k) A and B-patches as defined by [7].

For vertices v and w in V_1 and a nonnegative integer k (for our example we will let $v =$ vertex 1, $w =$ vertex 2, and $k = 3$), we define a (v, w, k) A-patch to be the sequence of edges

$$\epsilon_0 \epsilon_1 \epsilon_2 ... \epsilon_k \epsilon_{k+1}$$

in which

1. $\epsilon_0 \in \mathcal{E}$ and $\epsilon_{k+1} \in \mathcal{E}$
2. ϵ_0 begins at vertex v and ϵ_{k+1} ends at w
3. $\epsilon_i \in E_2$ for $i = 0, ...k$
4. If ϵ_i ends at vertex u, then ϵ_{i+1} begins at vertex u for $i = 0, ..., k$.

18a

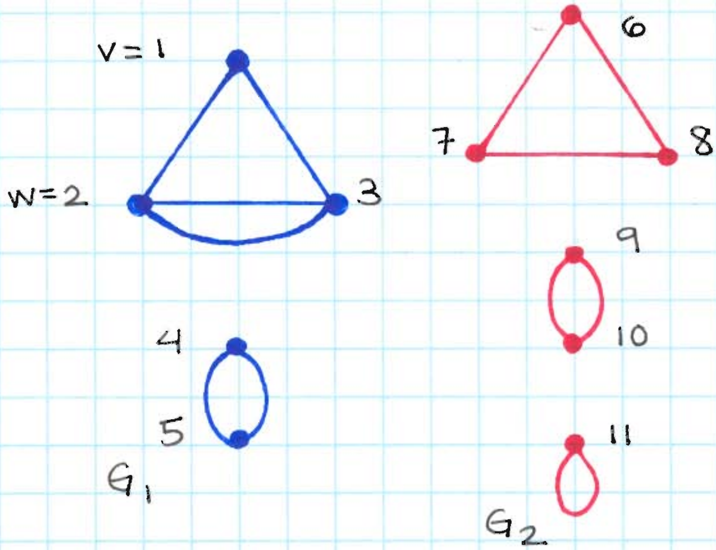

Figure 5.1

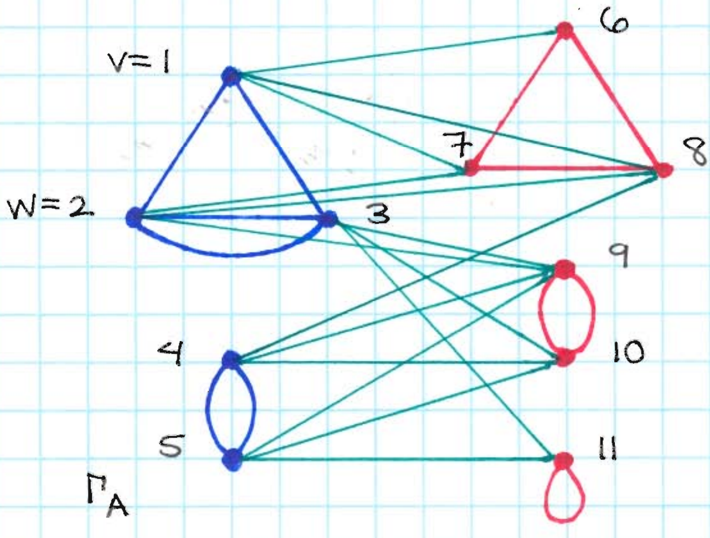

Figure 5.2

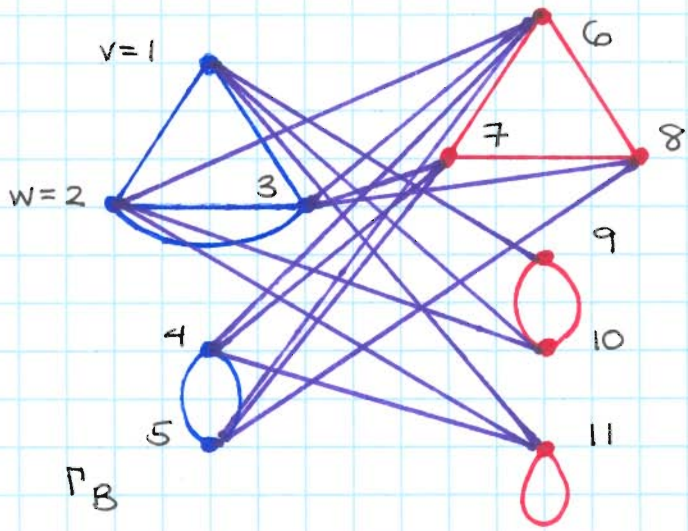

Figure 5.3

Thus, a (v, w, k) A-patch is a walk in Γ_A from v to w where only the first and last edges are in $\mathcal{E}$. We also have a (v, w, k) B-patch that is a walk from v to w in Γ_B where only the first and last edges are in $\mathcal{E}^{\mathcal{C}}$. We define this formally as a sequence of directed edges

$$\epsilon_0\epsilon_1\epsilon_2...\epsilon_k\epsilon_{k+1}$$

such that

1. $\epsilon_0 \in \mathcal{E}^{\mathcal{C}}$ and $\epsilon_{k+1} \in \mathcal{E}^{\mathcal{C}}$

2. ϵ_0 begins at vertex v and ϵ_{k+1} ends at w

3. $\epsilon_i \in E_2$ for $i = 0, ...k$

4. If ϵ_i ends at vertex u, then ϵ_{i+1} begins at vertex u for $i = 0, ..., k$.

We provide examples of both an A-patch and B-patch that can be seen in Figures 5.2 and 5.3 on pp.18a-b. For our example, we can see that the walk

$$1 \rightarrow 8 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 4$$

in Γ_A is an example of a $(1, 4, 3)$ A-patch. The walk

$$1 \rightarrow 11 \rightarrow (cw)11 \rightarrow (cw)11 \rightarrow (cw)11 \rightarrow 4$$

in Γ_B is an example of a $(1, 4, 3)$ B-patch. Here (cw) denotes clockwise movement around the loop at vertex 11.

We are then presented with the following lemma that is a significant part of the proof of the validity of Seidel switching:

Lemma 5.2. *Given a Seidel pair $(G_1, G_2, \mathcal{E})$, for each pair v and w of vertices in V_1 and each non-negative integer k, the number of (v, w, k) A-patches is equal to the number of (v, w, k) B-patches.*

To show this we are first required to show the following result.

Lemma 5.3. *Let G be an r-regular graph. Given a vertex v in G and a non-negative integer k, the number of k-walks in G which begin at v is r^k. Also, the number of k-walks in G which end at v is r^k.*

We can see this is true by examining our example, let $k = 3$. We thus move on to prove Lemma 5.2. We begin by considering vertices v and w in G_1 and partitioning the vertices of V_2 as follows:

$$V_2 = V_{(v,A)} \cup V_{(v,B)}$$

where $V_{(v,A)}$ is the set of vertices in V_2 adjacent to v in Γ_A and $V_{(v,B)}$ is the set of vertices in V_2 that are adjacent to v in Γ_B. We then create another partition in the same sense:

$$V_2 = V_{(w,A)} \cup V_{(w,B)}$$

where the vertices in $V_{(w,A)}$ are adjacent to w in Γ_A and the vertices in $V_{(w,B)}$ are adjacent to w in Γ_B. For our example, the partitions are as follows:

- $V_{(v,A)}$ = vertices 6, 7, and 8

- $V_{(v,B)}$ = vertices 9, 10 and 11

- $V_{(w,A)}$ = vertices 8, 9, and 10

- $V_{(w,B)}$ = vertices 6, 7, and 11

We can also note that the following holds for our example, and is equal to 3:

$$|V_{(v,A)}| = |V_{(v,B)}| = |V_{(w,A)}| = |V_{(w,B)}| = \frac{|V_2|}{2}.$$

We then partition all of the k-walks in G_2 into four sets,

$$W_{AA} \cup W_{AB} \cup W_{BA} \cup W_{BB}$$

according to where their beginning and ending vertices lie. This is done as follows:

- W_{AA} contains k-walks beginning in $V_{(v,A)}$ and ending in $V_{(w,A)}$.

- W_{AB} contains k-walks beginning in $V_{(v,B)}$ and ending in $V_{(w,A)}$.

- W_{BA} contains k-walks beginning in $V_{(v,A)}$ and ending in $V_{(w,B)}$.

- W_{BB} contains k-walks beginning in $V_{(v,B)}$ and ending in $V_{(w,B)}$.

For our example, we can now partition all 3-walks in G_2.

W_{AA}: We recall that these are the 3-walks beginning in $V_{(v,A)}$ and ending in $V_{(w,A)}$. Here we will have 3-walks beginning at vertices 6, 7, and 8 and ending at vertex 8, as listed below:

- $6 \to 8 \to 6 \to 8$
- $6 \to 8 \to 7 \to 8$
- $6 \to 7 \to 6 \to 8$
- $7 \to 8 \to 6 \to 8$
- $7 \to 8 \to 7 \to 8$
- $7 \to 6 \to 7 \to 8$
- $8 \to 6 \to 7 \to 8$
- $8 \to 7 \to 6 \to 8$

Here we can see that $|W_{AA}| = 8$.

W_{AB}: All 3-walks in G_2 beginning in $V_{(v,B)}$ and ending in $V_{(w,A)}$. The beginning vertices will be 9 and 10, and the ending vertices will be 9 and 10. Note that here we denote $9 \to (a)10$ and $9 \to (b)10$ to indicate the difference between the two edges connecting vertices 9 and 10.

- $9 \to (a)10 \to (b)9 \to (b)10$
- $9 \to (a)10 \to (b)9 \to (a)10$
- $9 \to (a)10 \to (a)9 \to (b)10$
- $9 \to (a)10 \to (a)9 \to (a)10$
- $9 \to (b)10 \to (b)9 \to (b)10$
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- $10 \to (b)9 \to (a)10 \to (b)9$

- $10 \to (b)9 \to (a)10 \to (a)9$

$|W_{AB}| = 16$

W_{BA}: We have all 3-walks in G_2 beginning in $V_{(v,A)}$ and ending in $V_{(w,B)}$. The beginning vertices will be 6, 7 and 8, and the ending vertices will be 6 and 7.

- $6 \to 8 \to 6 \to 7$
- $6 \to 8 \to 7 \to 6$
- $6 \to 7 \to 8 \to 6$
- $6 \to 7 \to 8 \to 7$
- $6 \to 7 \to 6 \to 7$
- $7 \to 8 \to 6 \to 7$
- $7 \to 8 \to 7 \to 6$
- $7 \to 6 \to 7 \to 6$
- $7 \to 6 \to 8 \to 6$
- $7 \to 6 \to 8 \to 7$
- $8 \to 6 \to 7 \to 6$

- $8 \to 6 \to 8 \to 6$
- $8 \to 6 \to 8 \to 7$
- $8 \to 7 \to 6 \to 7$
- $8 \to 7 \to 8 \to 6$
- $8 \to 7 \to 8 \to 7$

$|W_{BA}| = 16$

W_{BB}: All 3-walks from $V_{(v,B)}$ to $V_{(w,B)}$ beginning and ending at vertex 11. Here we use $11 \to (cw)11$ and $11 \to (ccw)11$ to denote the difference between clockwise and counterclockwise movement around the loop at vertex 11.

- $11 \to (cw)11 \to (cw)11 \to (cw)11$
- $11 \to (cw)11 \to (cw)11 \to (ccw)11$
- $11 \to (cw)11 \to (ccw)11 \to (cw)11$
- $11 \to (cw)11 \to (ccw)11 \to (ccw)11$
- $11 \to (ccw)11 \to (cw)11 \to (cw)11$
- $11 \to (ccw)11 \to (cw)11 \to (ccw)11$
- $11 \to (ccw)11 \to (ccw)11 \to (cw)11$

- $11 \to (ccw)11 \to (ccw)11 \to (ccw)11$

$|W_{BB}| = 8$

We can see that $W_{AA} \cup W_{AB}$ contains all k-walks in G_2 that begin at a vertex in $V_{(v,A)}$ and end in V_2. Since G_2 is r-regular, in the case of our example 2-regular, and there are exactly $\frac{|V_2|}{2}$, or 3, vertices in $V_{(v,A)}$, we apply Lemma 5.3 to get

$$|W_{AA}| + |W_{AB}| = |V_{(v,A)}|r^k = \frac{|V_2|}{2}r^k, \tag{5.1}$$

For our example,

$$8 + 16 = 3 * 2^3 = \frac{6}{2} * 2^3 = 32. \tag{5.2}$$

We also have that $W_{AB} \cup W_{BB}$ contains all of the k-walks, or 3-walks, in G_2 beginning in V_2 and ending at a vertex in $V_{(w,B)}$. This gives us

$$|W_{AB}| + |W_{BB}| = |V_{(w,B)}|r^k = \frac{|V_2|}{2}r^k. \tag{5.3}$$

Thus, we have,

$$16 + 8 = 3 * 2^3 = \frac{6}{2} * 2^3 = 32. \tag{5.4}$$

From (5.1) and (5.3), we have

$$|W_{AA}| + |W_{AB}| = |W_{AB}| + |W_{BB}|, \tag{5.5}$$

and it follows that

$$|W_{AA}| = |W_{BB}|.$$

We can see that these equalities hold for our example.

To finish the proof of Lemma 5.2, we can see that each 3-walk in W_{AA} corresponds to exactly one $(v, w, 3)$ A-patch, and each 3-walk in W_{BB} corresponds to exactly one $(v, w, 3)$ B-patch.

The validity of the following corollary is a result of W_{AA} and W_{BB} having the same cardinality.

Corollary 5.4. *Let v and w be vertices in V_1 and k a non-negative integer. Let $\mathcal{P}^A_{(v,w,k)}$ be the set of (v, w, k) A-patches from v to w and $\mathcal{P}^B_{(v,w,k)}$ be the set of (v, w, k) B-patches from v to w. Then there is a one-to-one correspondence*

$$\mathcal{P}^A_{(v,w,k)} \leftrightarrow \mathcal{P}^B_{(v,w,k)}.$$

We can now show that Γ_A and Γ_B of a Seidel pair are length isospectral.

Theorem 5.5. *Let k be a non-negative integer and $(G_1, G_2, \mathcal{E})$ a Seidel pair. There is a one-to-one correspondence between the set of all closed k-walks in Γ_A and the set of all closed k-walks in Γ_B.*

We divide every closed k-walk W in Γ_A, recall Figure 6.5, into the following three types:

- Type I: W is contained in G_1

- Type II: W is contained in G_2
- Type III: W contains some edges in $\mathcal{E}$.

We do the same for all closed k-walks Z in Γ_B, recall Figure 6.6:

- Type I$'$: Z is contained in G_1
- Type II$'$: Z is contained in G_2
- Type III$'$: Z contains some edges in $\mathcal{E}^{\mathcal{C}}$.

For a comprehensive list of all such walks in our example, see Section A of the Appendix.

For a closed k-walk, or 3-walk, W in Γ_A, if W is Type I or Type II, then it is also a closed k-walk, or 3-walk, of Type I$'$ or Type II$'$ in Γ_B. Thus we have a correspondence for these walks via the identity mapping.

The completion of the proof requires the construction of a bijection from Type III closed k-walks in Γ_A to Type III$'$ closed k-walks in Γ_B. Let

$$W = \epsilon_1\epsilon_2...\epsilon_k$$

be a Type III closed k-walk in Γ_A. For our example, we have

$$W = \epsilon_1\epsilon_2\epsilon_3.$$

Consider

$$W = \{3,1\}, \{1,7\}, \{7,3\}.$$

Since W is closed and contains some edge in $\mathcal{E}$, it contains some edge ϵ_m in $\mathcal{E}$ that goes from a vertex in V_1 to a vertex in V_2. By permuting the edges in W, we obtain

$$\overline{W} = \epsilon_m \epsilon_{m+1} \epsilon_{m+2} \ldots \epsilon_{m+k-1}$$

where we read the subscripts modulo k. This is a closed k-walk beginning and ending at some vertex v in V_1. For our example, we have

$$\overline{W} = \epsilon_m \epsilon_{m+1} \epsilon_{m+2}$$

and

$$\overline{W} = \{1,7\}, \{7,3\}, \{3,1\},$$

a closed 3-walk beginning and ending at vertex 1.

We can view $\overline{W}$ as the following sequence:

$$\overline{W} = P_1 W_1 P_2 W_2 \ldots P_j W_j$$

where each P_i is a (v_i, w_i, l_i) A-patch and each W_i is a walk in G_1 from w_i to v_{i+1}, with subscripts modulo j. For our example, we have

$$\overline{W} = P_1 W_1$$

such that

$$P_1 = (1,3,2) \ A\text{-patch passing through vertex 7 in } \Gamma_A$$

and

$$W_1 = \text{a walk of length 1 from vertex 3 to 1.}$$

By Corollary 5.4, we have a (v_i, w_i, l_i) B-patch P_i' that corresponds to each P_i patch. Thus, we can replace each P_i in $\overline{W}$ with its corresponding P_i' to get a new walk

$$\overline{W}' = P_1'W_1P_2'W_2...P_j'W_j.$$

For our example, we have

$$P_1' = (1,3,2) \text{ B-patch passing through vertex 11 in } \Gamma_B$$

and

$$\overline{W}' = P_1'W_1.$$

We now have a closed k-walk, or 3-walk, beginning and ending at v = vertex 1. However, each ϵ_h of $\overline{W}$ contained in $\mathcal{E}$ has been replaced by ϵ_h' in $\mathcal{E}^{\mathcal{C}}$. Therefore, $\overline{W}$ is a closed k-walk in Γ_B. To complete this mapping, we take

$$\overline{W}' = \epsilon_m'\epsilon_{m+1}'\epsilon_{m+2}'...\epsilon_{m+k-1}'$$

such that each $\epsilon_h' = \epsilon_h$ or its corresponding edge in some P_i'. Thus, we have

$$\overline{W}' = \epsilon_m'\epsilon_{m+1}'\epsilon_{m+2}'$$

such that $\epsilon'_m = \{1, 11\}, \epsilon'_{m+1} = \{11, 3\}$, and $\epsilon'_{m+2} = \{3, 1\} = \epsilon_{m+2}$.

We obtain a Type III$'$ closed 3-walk in Γ_B by applying another cyclic permutation to the edges in $\overline{W}$

$$W' = \epsilon'_1\epsilon'_2...\epsilon'_k$$

$$W' = \epsilon'_1\epsilon'_2\epsilon'_3$$

$$W' = \{3, 1\}, \{1, 11\}, \{11, 3\}$$

We may invert this mapping $W \mapsto W'$ as follows: a closed 3-walk W' in Γ_B contains some first edge ϵ'_m in $\mathcal{E}$ from V_1 to V_2.

$$W' = \{3, 1\}, \{1, 11\}, \{11, 3\}$$

such that $\epsilon'_m = \{1, 11\}$ in $\mathcal{E}^{\mathcal{C}}$ from vertex 1 of G_1 to vertex 11 of G_2. We then apply a cyclic permutation to the edges in W' such that ϵ'_m is first

$$\overline{W}' = \{1, 11\}, \{11, 3\}, \{3, 1\}$$

and replace each (v_i, w_i, l_i) B-patch with its corresponding A-patch which is known to exist by Corollary 5.4.

$$P'_1 = (1, 3, 2) \text{ } A\text{-patch through vertex 11}$$

$$\downarrow$$

$$P'_1 = (1, 3, 2) \text{ } B\text{-patch through vertex 7.}$$

Undoing the cyclic permutation so that the image of ϵ_1' is first is our last step

$$W = \{3, 1\}, \{1, 7\}, \{7, 3\}.$$

We may verify this by observing the following of our example:

- The first edge in W from V_1 to V_2 is in the same position as the first edge in W' from V_1 to V_2.
- Replacing a B-patch with an A-patch is the inverse of replacing an A-patch with a B-patch.

6 Seidel switching and regularity

We shall now look at Seidel switching with a further restriction on the construction. This is done because, although one might intuitively think the opposite is true, cospectral non-isomorphic pairs of regular graphs appear to be rarer than cospectral pairs of non-regular graphs. Thus we look at Seidel switching under the following restrictions for a regular Seidel pair provided in [7] p.14.

Theorem 6.1. *Suppose* $(G_1, G_2, \mathcal{E})$ *is a Seidel pair, and that* Γ_A *is* $q-$*regular. Then*

1. $|V_1|$ *is even.*

2. G_1 *is regular.*

3. $\frac{|V_1|}{2} + r = \frac{|V_2|}{2} + s = q$*, where* r *is the valency of* G_2 *and* s *is the valency of* G_1*.*

4. Γ_B *is* $q-$*regular.*

We begin by providing an example of a regular construction. Figures 6.1-6.3 on p.33a show G_1, G_2, and a pair of isomorphic Γ_A and Γ_B that fit the conditions of Theorem 6.1. One can refer back to this example while working through

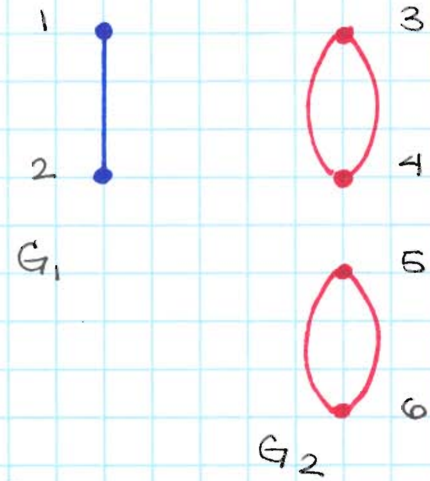

Figure 6.1

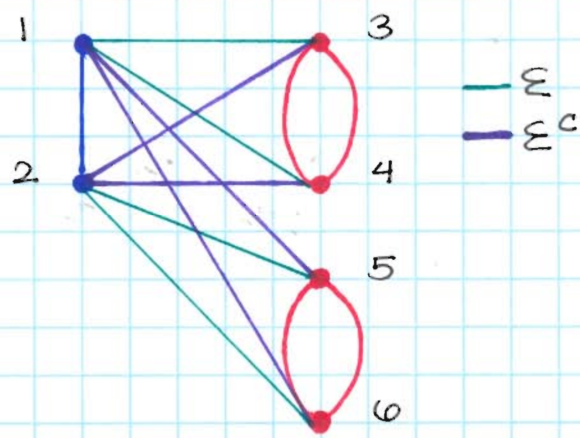

Figure 6.2: G_1 and G_2 with $\mathcal{E}$ and $\mathcal{E}^c$

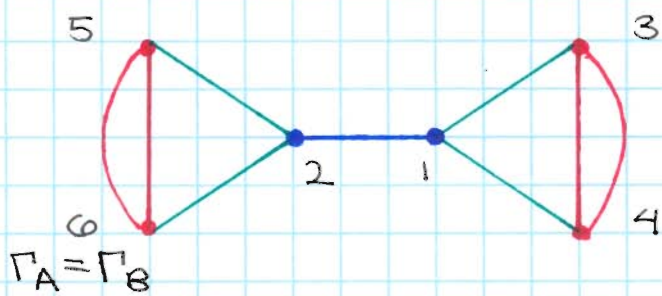

Figure 6.3: Γ_A, Γ_B regular

the following proof. A proof of Theorem 6.1 similar to the following can be found in [7].

Proof. As in all cases of Seidel switching, G_2 is r-regular, so in Γ_A each vertex from V_2 will be an endpoint of the same number of edges in $\mathcal{E}$. This number will be $\frac{|\mathcal{E}|}{|V_2|}$. This is because in $\mathcal{E}$ edges will be distributed evenly for all $v_i \in V_2$. Using the previously described method of Seidel construction, we know that

$$|\mathcal{E}| = |V_1| \times \frac{|V_2|}{2} = \frac{|V_1| \times |V_2|}{2}.$$

So, the number of edges in $\mathcal{E}$ at each vertex in G_2 will be $\frac{|V_1|}{2}$. This indicates that $|V_1|$ must be even and

$$q = r + \frac{|V_1|}{2}. \tag{6.1}$$

We now consider a vertex v of G_1 with a valency of s. After constructing Γ_A there are $\frac{|V_2|}{2}$ new edges adjacent to v. This follows from the reasoning used above for G_2. Thus, the valency of v as a vertex of Γ_A is

$$s + \frac{|V_2|}{2}.$$

We know that Γ_A is $q-$regular. So, we can see that

$$s = q - \frac{|V_2|}{2},$$

for each choice of v. Thus, we have that G_1 is regular and

$$q = s + \frac{|V_2|}{2}. \tag{6.2}$$

With this we have proved the third part of Theorem 6.1 in (6.1) and (6.2).

We now use a similar argument to prove that Γ_B is q-regular.

Using the fact that $\mathcal{E}^{\mathcal{C}}$ is made up of edges that join each vertex of the graph G_1 to exactly half of the vertices of the graph G_2, it is clear that the valency of each vertex in the set V_1 in the graph Γ_B is

$$s + \frac{|V_2|}{2} = q.$$

Now let v be a vertex of G_2. We have $\frac{|V_1|}{2}$ vertices in V_1 that are adjacent to v via edges in $\mathcal{E}$. Thus, there are also $\frac{|V_1|}{2}$ vertices in V_1 that are not adjacent to v via edges in $\mathcal{E}$. These vertices make up the set of vertices that will be connected to v by edges in the set $\mathcal{E}^{\mathcal{C}}$. Therefore, the valency of a vertex v in the graph Γ_B is

$$r + \frac{|V_1|}{2} = q.$$

□

Definition 6.2. *When all of the conditions of Theorem 6.1 are met, we will refer to* $(G_1, G_2, \mathcal{E})$ *as a* ***regular Seidel pair****.*

In [7], Quenell indicates without example that conditions 1, 2 and 3 of Theorem 6.1 are not sufficient for the regularity of Γ_A and Γ_B. Here we provide an example illustrating this insufficiency. Figure 6.4 on p.35a gives G_1 and G_2

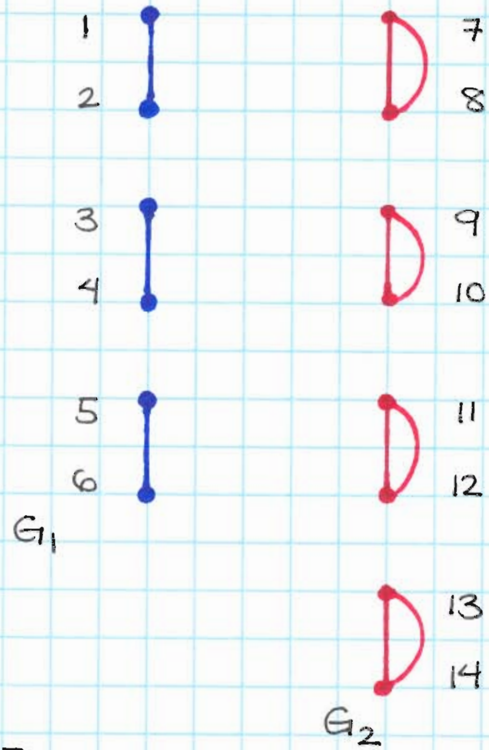

Figure 6.4

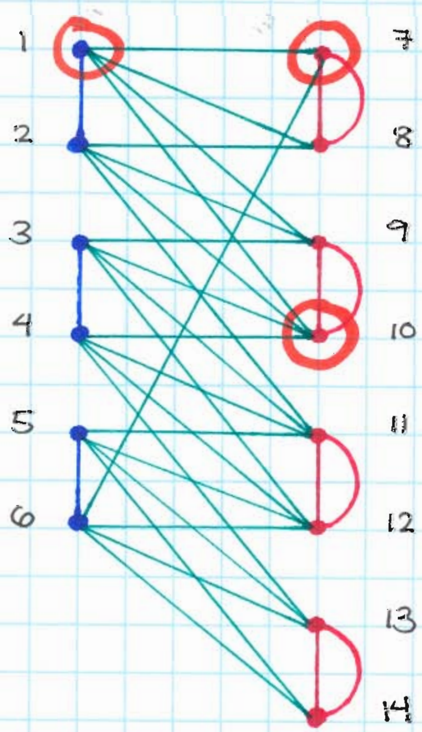

Figure 6.5: Insufficiency of conditions 1, 2, 3 of Theorem 6.1

such that G_1 has an even number of vertices, 6, and is 1-regular. We also see that $\frac{|V_1|}{2}+r=3+2=5$ and $\frac{|V_2|}{2}+s=4+1=5$, thus satisfying conditions 1, 2, and 3 of Theorem 6.1. However, once we add $\mathcal{E}$ as shown in Figure 6.5 on p.35a, we generate Γ_A and Γ_B that are non-regular. To see this, compare vertices 1, 7, and 10. We note that there do exist $\mathcal{E}$ that would induce regularity, but it is not guaranteed that such a $\mathcal{E}$ will be chosen.

Corollary 6.3. *When the conditions of Theorem 6.1 are met, $|V_1| = |V_2|$ if and only if $r = s$.*

Proof. Consider

$$\frac{|V_1|}{2} + r = \frac{|V_2|}{2} + s$$

Then we have

$$|V_1| = |V_2|$$

$$\Leftrightarrow$$

$$\frac{|V_1|}{2} + r = \frac{|V_1|}{2} + s$$

$$\Leftrightarrow$$

$$r = s$$

□

Corollary 6.4. *If $(G_1, G_2, \mathcal{E})$ is a Seidel pair in which either G_1 or G_2 contains only two vertices, then Γ_A is isomorphic to Γ_B.*

Quenell indicates in [7], without proof, that using the above and "a little more work", the following can be shown using the idea that G_1 must be the union of two K_2s and the automorphism group of G_1 is large enough to induce an isomorphism between Γ_A and Γ_B for any choice of $\mathcal{E}$.

Corollary 6.5. *If* $|V_1| = |V_2| = 4$ *and* G_1 *is 1-regular, then the graphs* Γ_A *and* Γ_B *in any Seidel pair* $(G_1, G_2, \mathcal{E})$ *are isomorphic.*

We are able to give a proof as follows:

Proof. Corollary 5.2 implies that G_2 must also be 1-regular.

We now look to give a concrete proof. We first notice that Theorem 5.1, Corollary 5.2 and our hypothesis imply that G_2 must also be the union of two K_2s. See Figure 6.6 on p.37a.

We provide four different choices of $\mathcal{E}$ and consider the mapping of vertices in V_A to vertices in V_B. Figures 6.7-6.18 on pp.37a-d show us G_1 and G_2 as well as these four different examples of $\mathcal{E}$ that generate pairs of isomorphic graphs. Note the differences in location of vertices 1, 2, 3, and 4 in each Γ_A and Γ_B.

We determine the automorphism group of G_1 as follows:

- $\sigma_0 : 1\ 2\ 3\ 4$ (1)

- $\sigma_1 : 1\ 2\ 4\ 3$ (34)

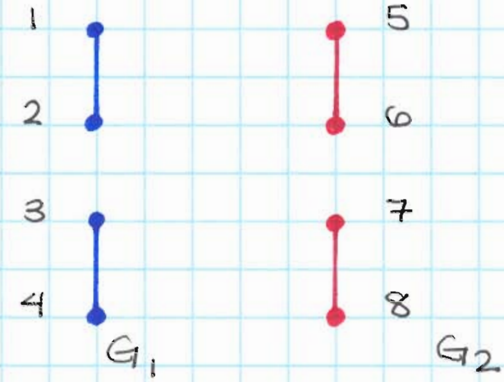

Figure 6.6

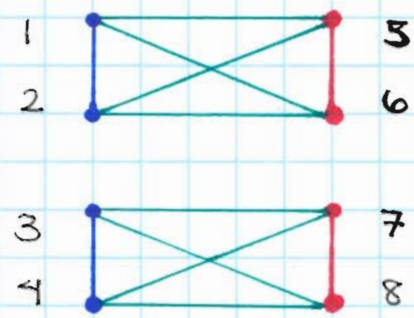

Figure 6.7: G_1 and G_2 with first choice of $\mathcal{E}$

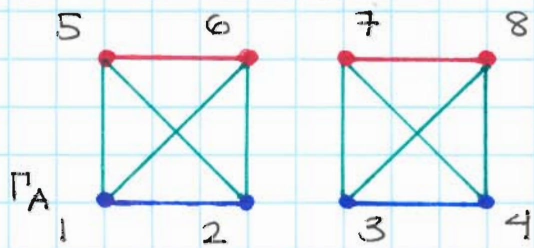

Figure 6.8: Γ_A for first choice of $\mathcal{E}$

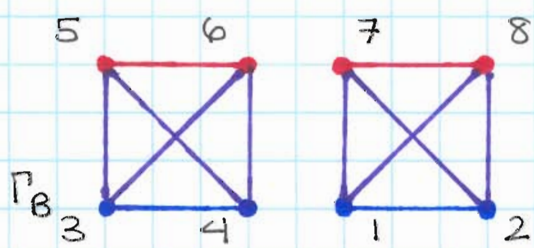

Figure 6.9: Γ_B for first choice of $\mathcal{E}$

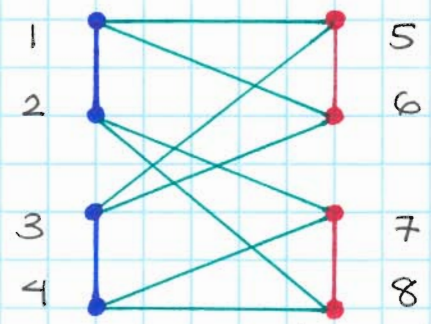

Figure 6.10: G_1 and G_2 with second choice of $\mathcal{E}$

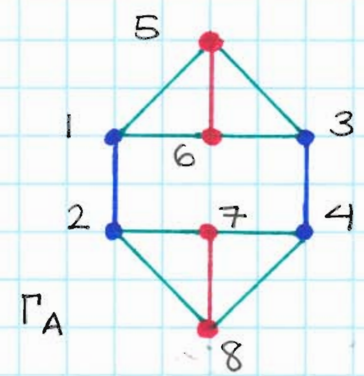

Figure 6.11: Γ_A for second choice of $\mathcal{E}$

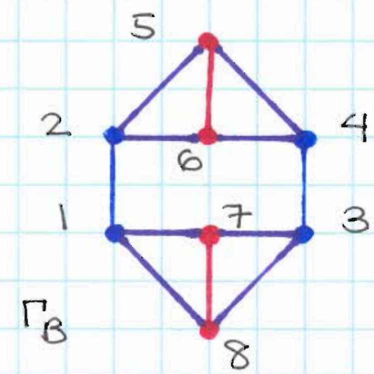

Figure 6.12: Γ_B for second choice of $\mathcal{E}$

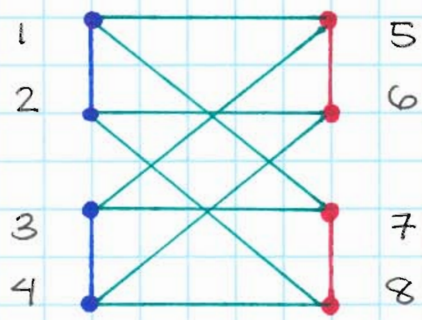

Figure 6.13: G_1 and G_2 with third choice of $\mathcal{E}$

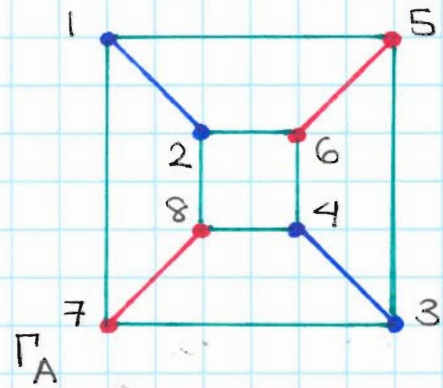

Figure 6.14: Γ_A for third choice of $\mathcal{E}$

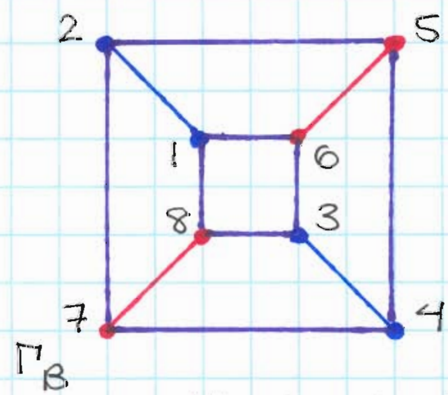

Figure 6.15: Γ_B for third choice of $\mathcal{E}$

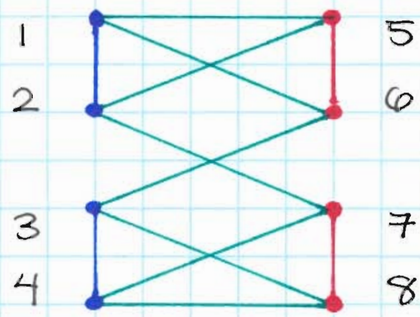

Figure 6.16: G_1 and G_2 with fourth choice of $\mathcal{E}$

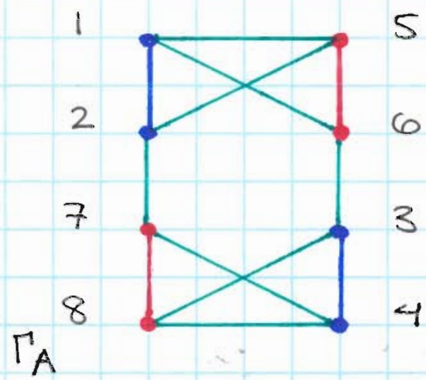

Figure 6.17: Γ_A for fourth choice of $\mathcal{E}$

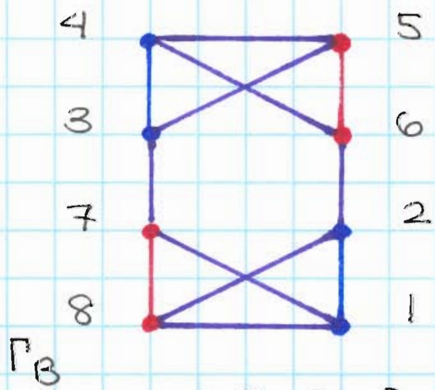

Figure 6.18: Γ_B for fourth choice of $\mathcal{E}$

- $\sigma_2 : 2\ 1\ 3\ 4$ (12)

- $\sigma_3 : 2\ 1\ 4\ 3$ (12)(34)

- $\sigma_4 : 3\ 4\ 2\ 1$ (13)(24)

- $\sigma_5 : 3\ 4\ 1\ 2$ (1324)

- $\sigma_6 : 4\ 3\ 1\ 2$ (1423)

- $\sigma_7 : 4\ 3\ 2\ 1$ (14)(23)

Thus,

$$Aut(G_1) = \{1, (12), (34), (12)(34), (13)(24), (1324), (1423), (14)(23)\}.$$

This is D_4, a subgroup of S_4, corresponding to the symmetries of the square as shown in Figure 6.20 on p.38a.

We represent the vertices of G_1 as the vertices of a square, as shown in Figure 6.19 on p.38a. We have two adjacent pairs corresponding to two diagonals of the square and four non-adjacent pairs corresponding to the four sides of the square.

We note that $\mathrm{Aut}(G_1)$ is transitive on the two adjacent pairs and the four nonadjacent pairs. In fact, the following is clear:

Proposition 6.6. *For vertices $\{a, b, c, d\}$ of G_1,*

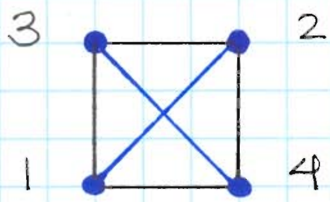

Figure 6.19

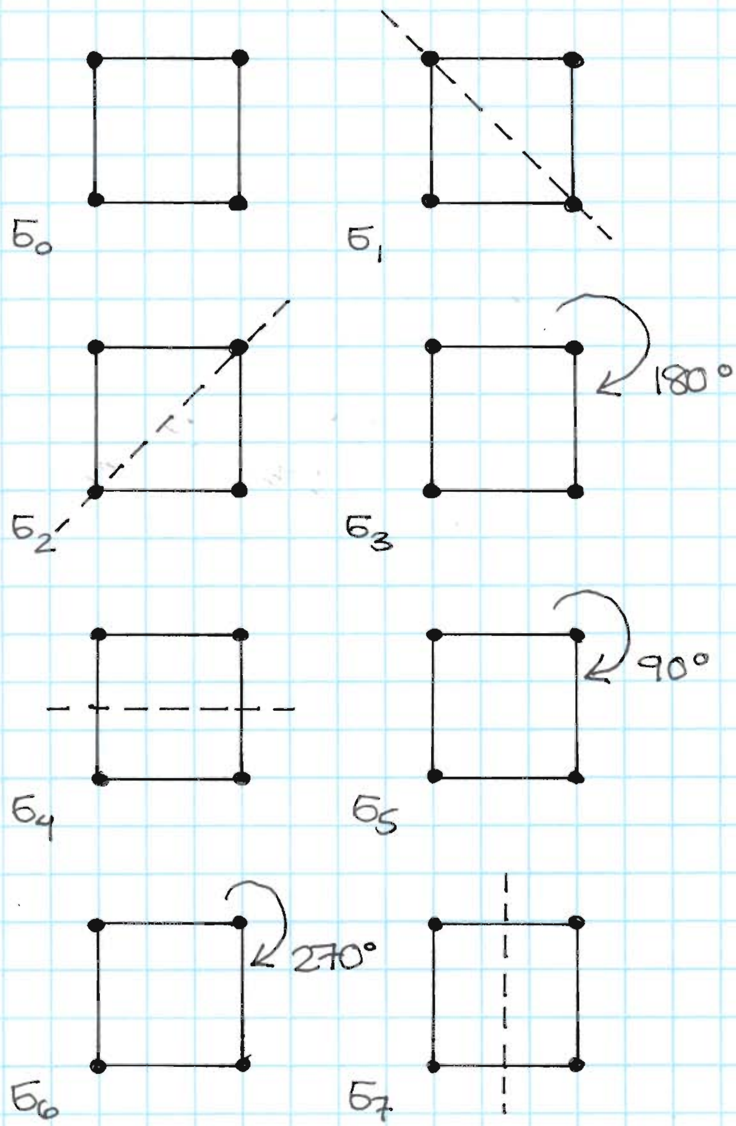

Figure 6.20: $\mathrm{Aut}(G_1)$ is the symmetries of the square, D_4.

1. *If a and d are adjacent and b and c are adjacent in G_1, then $(ad)(bc) \in Aut(G_1)$.*

2. *If a and d are nonadjacent and b and c are nonadjacent in G_1, then $(ad)(bc) \in Aut(G_1)$.*

Given G_1 and G_2 as shown and labeled earlier, we let

$$\text{vertices of } G_1 = \{1, 2, 3, 4\} = \{a, b, c, d\}$$

$$\text{vertices of } G_2 = \{5, 6, 7, 8\} = \{r, s, t, u\}$$

and think of the choice of $\mathcal{E}$ as a map

$$\phi : \{1, 2, 3, 4\} \to \text{pairs}\{56, 57, 58, 67, 68, 78\}$$

and $\mathcal{E}^{\mathcal{C}} \leftrightarrow \phi^C$.

We can divide these mappings into two cases, $|\phi^{-1}\{r, s\}| = 2$ for some pair $\{r, s\}$ and $|\phi^{-1}\{r, s\}| \leq 1$ for all pairs $\{r, s\}$.

Case 1:$|\phi^{-1}\{r, s\}| = 2$ for some pair $\{r, s\}$. This forces the following:

ϕ	Γ_A	Γ_B
$\phi(a) = \{r, s\}$	ar,as$\in \mathcal{E}$	at,au$\in \mathcal{E}^{\mathcal{C}}$
$\phi(b) = \{r, s\}$	br,bs$\in \mathcal{E}$	bt,bu$\in \mathcal{E}^{\mathcal{C}}$
$\phi(c) = \{t, u\}$	ct,cu$\in \mathcal{E}$	cr,cs$\in \mathcal{E}^{\mathcal{C}}$
$\phi(d) = \{t, u\}$	dt,du$\in \mathcal{E}$	dr,ds$\in \mathcal{E}^{\mathcal{C}}$

If a and b are adjacent, then c and d are also adjacent and if a and b are nonadjacent, then c and d are also nonadjacent. We can say that there exists $\sigma \in Aut(G_1)$ such that $\sigma\{a,b\} = \{c,d\}$. This σ induces $\Gamma_A \simeq \Gamma_B$.

Case 2: $|\phi^{-1}\{r,s\}| \leq 1$ for all pairs $\{r,s\}$.

Suppose

$$\phi(a) = \{r,s\}.$$

Then,

$$\phi(b) = \{r,t\}$$

$$\phi(c) = \{s,u\}$$

$$\phi(d) = \{t,u\}$$

because r and s cannot occur together again, so t and u must occur in combination with r and s in some order. Thus, we have the following:

Γ_A	Γ_B
ar,as$\in \mathcal{E}$	at,au$\in \mathcal{E}^{\mathcal{C}}$
br,bt$\in \mathcal{E}$	bs,bu$\in \mathcal{E}^{\mathcal{C}}$
cs,cu$\in \mathcal{E}$	cr,ct$\in \mathcal{E}^{\mathcal{C}}$
dt,du$\in \mathcal{E}$	dr,ds$\in \mathcal{E}^{\mathcal{C}}$

We look at Figures 6.21 and 6.22 on p.40a to see the differences in $\mathcal{E}$ and $\mathcal{E}^{\mathcal{C}}$.

Claim: $\sigma = (ad)(bc) \in Aut(G_1)$ induces $\Gamma_A \simeq \Gamma_B$.

Case A: If a and d are adjacent then b and c are adjacent. This can be seen in Figure 6.23 on p.40b.

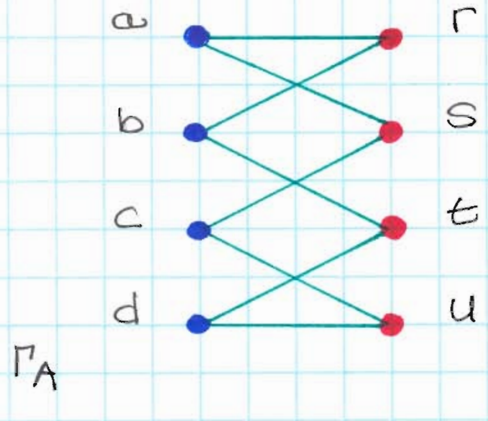

Figure 6.21: $\mathcal{E}$ for case 2

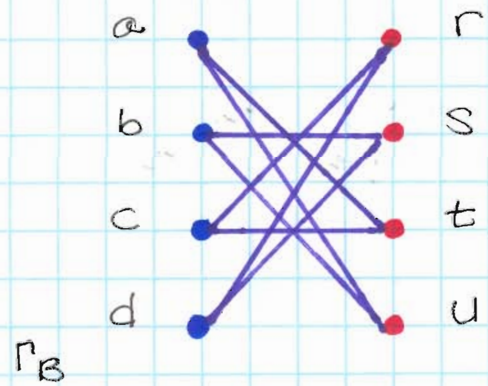

Figure 6.22: $\mathcal{E}^c$ for case 2

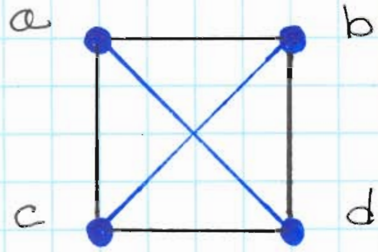

Figure 6.23: Case A

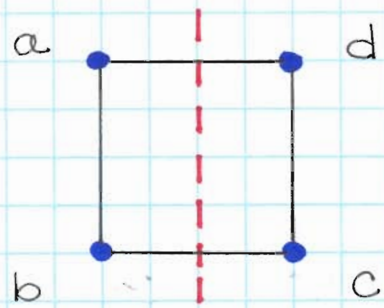

Figure 6.24: Possibility(i) of case B

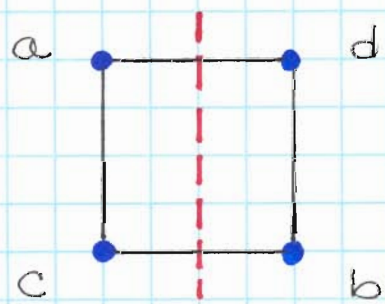

Figure 6.25: Possibility(ii) of case B

Case B: If a and d are nonadjacent then b and c are nonadjacent. There are two possibilities for this case, as shown in Figures 6.24 and 6.25 on p.40b. $\square$

After looking at various examples of Seidel pairs for different combinations of G_1 and G_2, the following conjecture emerged.

Conjecture: *If G_1, G_2 are both regular and $|V_1|$ and $|V_2|$ are both even, then Γ_A and Γ_B will both be regular.*

We look to Figures 6.26-6.28 on p.41a for an example illustrating the ideas of this conjecture.

There are, however, a few questions that arise from this conjecture:

1. Do G_1 and G_2 both need to have the same number of vertices?

2. Do G_1 and G_2 need to have the same valency?

We now attempt to answer these questions in an effort to support the conjecture and determine if it is indeed true.

1. The example shown in Figures 6.29 and 6.30 on p.41b shows that when G_1 and G_2 have a different number of vertices, our conjecture does not hold. This is clear from comparing the degrees of vertices 4 and 10. This counterexample shows us that the original conjecture is false, indicating a need for further

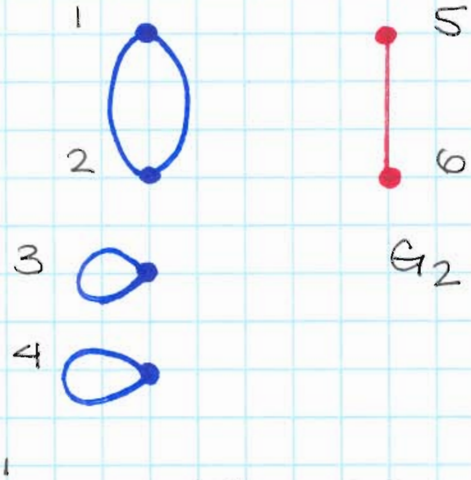

Figure 6.26: G_1 and G_2

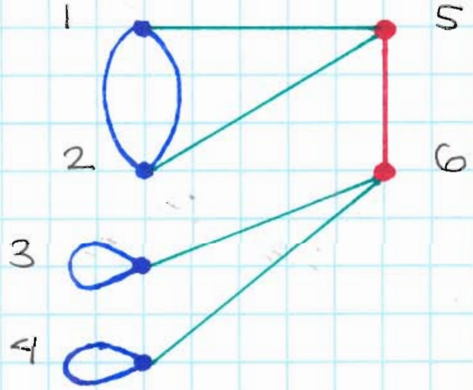

Figure 6.27: G_1 and G_2 with $\mathcal{E}$

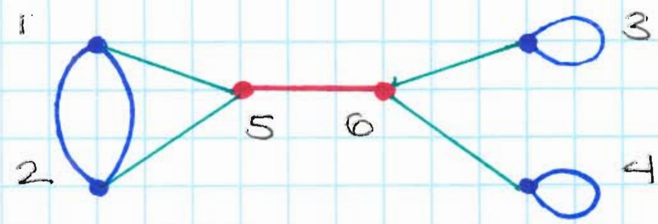

Figure 6.28: Γ_A for general example of conjecture.

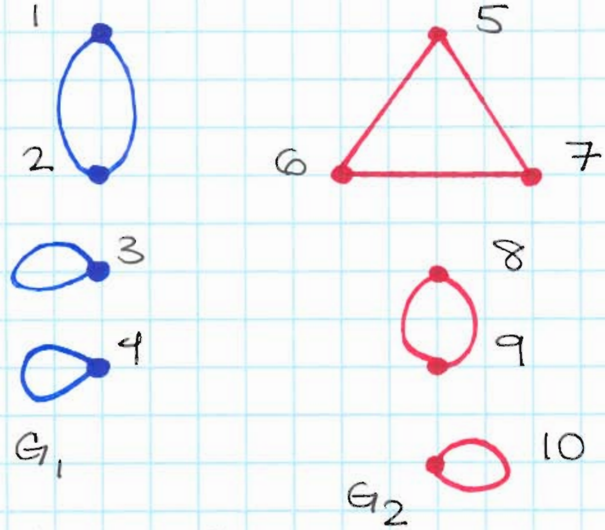

Figure 6.29

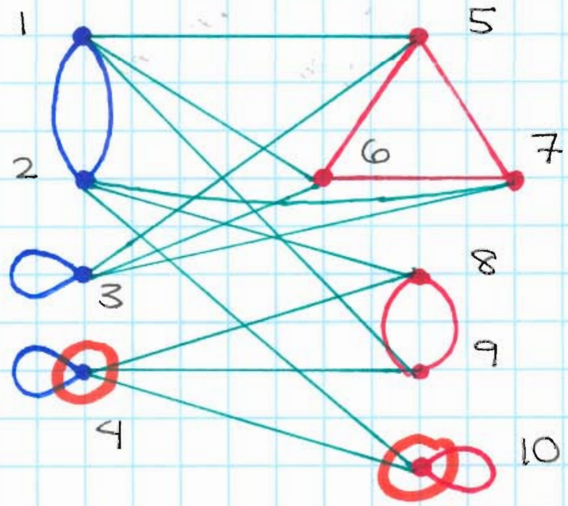

Figure 6.30: G_1 and G_2 with E for condition 1

restriction on the hypothesis. Thus, we now provide a revised version of the conjecture to account for what we have just learned:

Conjecture (Revised): *If G_1, G_2 are both regular and $|V_1| = |V_2| = 2N$, then Γ_A, Γ_B will both be regular.*

2. Figures 6.31-6.33 on p.42a give us a pair of G_1 and G_2 with different valencies. This indicates to us that the second condition we are questioning is not necessary. We have been able to prove a special case of the conjecture and also show that the resulting Γ_A and Γ_B are cospectral. The following theorem illustrates an example of the conjecture with the two mentioned possible restrictions.

Theorem 6.7. *Given q-regular G_1 and G_2 such that*

$$|V_1| = |V_2| = 2N.$$

Then

1. *Γ_A and Γ_B are $(q+N)$-regular.*

2. *Γ_A and Γ_B are cospectral.*

Proof. Here we provide the proof of (1).

We are given the following:

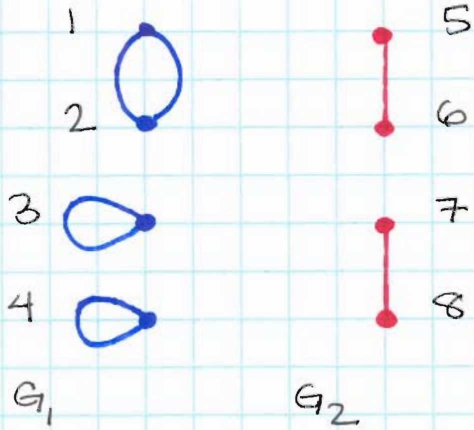

Figure 6.31

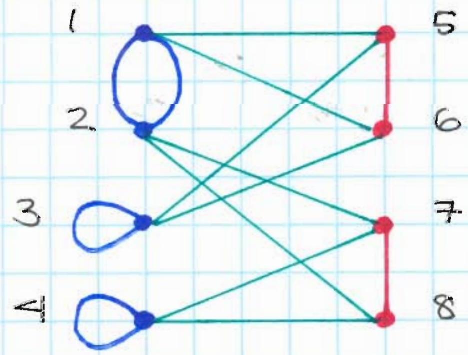

Figure 6.32: G_1 and G_2 with $\mathcal{E}$

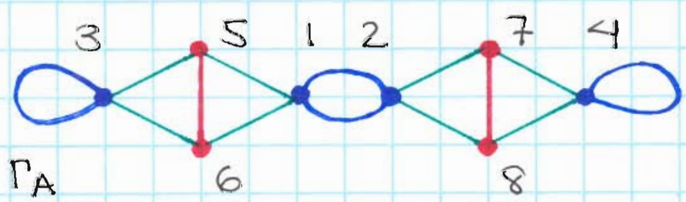

Figure 6.33: Γ_A for condition 2

$$|V_1| = |V_2| = 2N$$

$$\Gamma_A = (V_A, E_A) = (V_1 \cup V_2, E_1 \cup E_2 \cup \mathcal{E})$$

$$\Gamma_B = (V_B, E_B) = (V_1 \cup V_2, E_1 \cup E_2 \cup \mathcal{E}^{\mathcal{C}})$$

$$|V_A| = |V_B| = 4N$$

$$|\mathcal{E}| = |\mathcal{E}^{\mathcal{C}}|$$

Since $\mathcal{E}$ is a set of edges connecting half of the vertices in G_1 to every vertex in G_2, we are adding N edges to each vertex upon construction of Γ_A. The same occurs with the construction of Γ_B using $\mathcal{E}^{\mathcal{C}}$. Given that G_1 and G_2 are q-regular, we can now see that Γ_A and Γ_B are $(q + N)$-regular.

We refer you to section 5 for an illustrated example that works through the proof provided in [7] of the fact that Γ_A and Γ_B will be cospectral for any Seidel switch; this is a special case. $\square$

This is also a special case of the regular Seidel pair described in Theorem 6.1.

7 Seidel switching and the Ihara zeta function

We now return to the Ihara zeta function defined earlier and discuss a connection between Seidel switching and two graphs with equal Ihara zeta functions.

Ihara defined the zeta function as a product over p-adic group elements. It was not until Serre looked at the product that the Ihara zeta function had a graph theoretic interpretation. Sunada, Hashimoto, Bass and others extended the theory.

Through the following theorem, Aubi Mellien was able to show a direct connection between the equality of Ihara zeta functions of two graphs and their cospectrality. Mellien, a student participating in the mathematics REU at LSU during the summer of 2001, developed this theorem while working with Robert Perlis.

Theorem 7.1. ***Aubi's Theorem:*** *Let Γ, Γ' be regular md2 graphs. Then $Z_\Gamma(u) = Z_{\Gamma'}(u)$ if and only if Γ and Γ' are cospectral.*

A proof of Aubi's Theorem can be found on pp.22-24 of [2] and is referred to as Theorem 3.1.2.

The next theorem follows from Aubi's Theorem, as stated above, and the proven fact that Γ_A and Γ_B are cospectral (Theorem 5.1).

Theorem 7.2. *For q-regular graphs G_1 and G_2 such that $q \geq 1$, $Z_{\Gamma_A}(u) = Z_{\Gamma_B}(u)$ where (Γ_A, Γ_B) is any Seidel switch of G_1, G_2.*

8 Small, regular Seidel pairs

One aspect of interest in regards to Seidel switching is the construction of regular Seidel pairs such that Γ_A and Γ_B are non-isomorphic. It is of particular interest to look at small examples. From [7],

Lemma 8.1. *There exist at least three regular Seidel pairs with Γ_A not isomorphic to Γ_B in which*

$$|V_1| = |V_2| = 4$$

and G_1 and G_2 are both 2-regular.

In [7] (p.17), Quenell provides one example of such a Seidel pair. Here we provide two more examples of Seidel pairs of this type.

Figures 8.1-8.3 on p.45a show us the Seidel pair given in [7]. Figures 8.4-8.9 on p.45b-c then provide the two more non-isomorphic Seidel pairs we have found such that the described conditions have been met.

Thus we can now see that there are at least three regular Seidel pairs of this type.

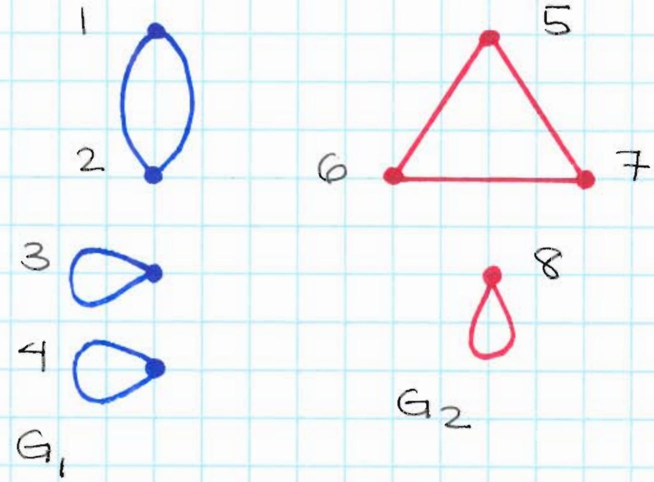

Figure 8.1 (example 1)

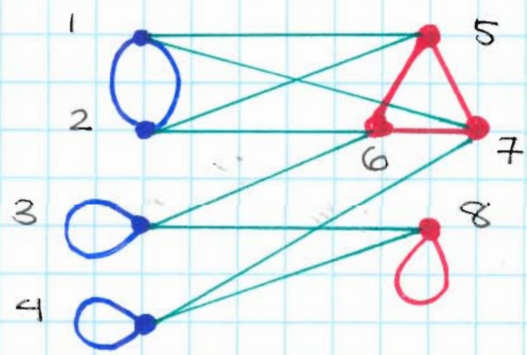

Figure 8.2: G_1 and G_2 with $\mathcal{E}$ (example 1)

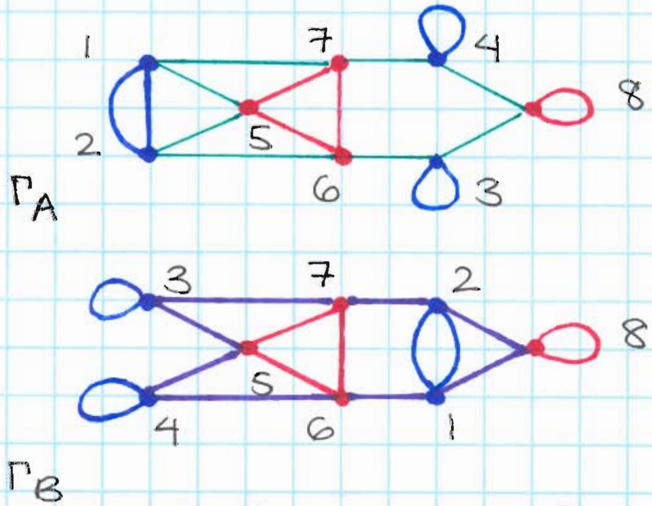

Figure 8.3: First example of Lemma 8.1 [6]

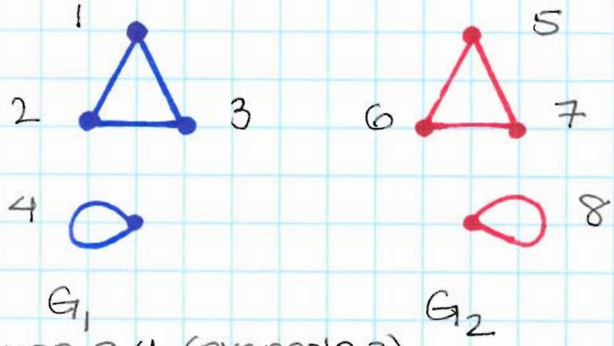

Figure 8.4 (example 2)

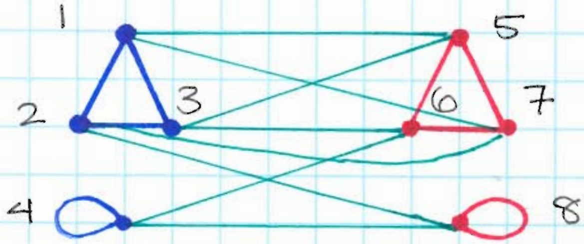

Figure 8.5: G_1 and G_2 with $\mathcal{E}$ (example 2)

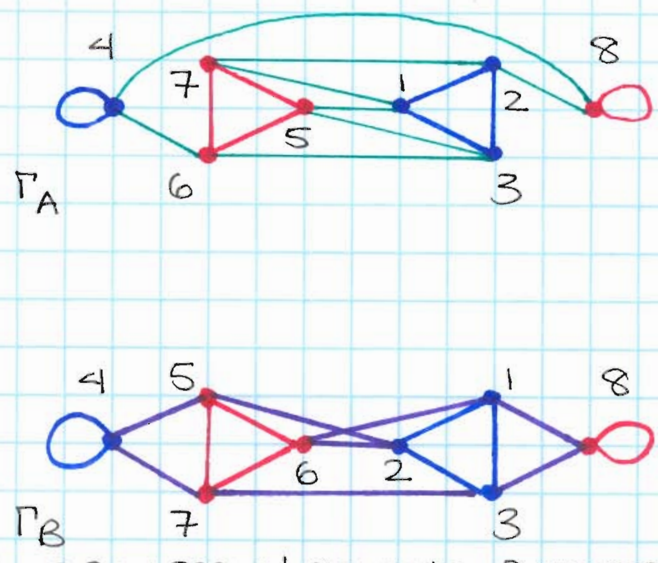

Figure 8.6: Second example of Lemma 8.1

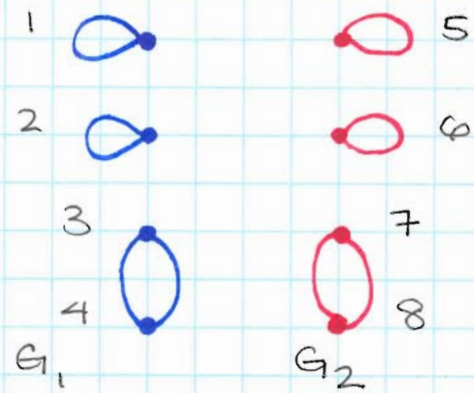

Figure 8.7 (example 3)

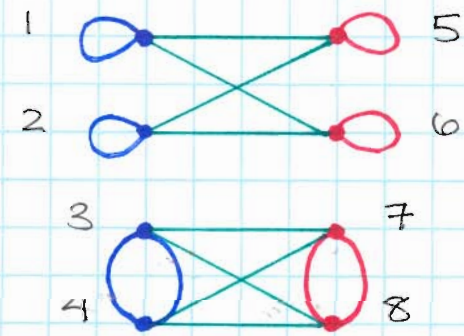

Figure 8.8 : G_1 and G_2 with $\mathcal{E}$ (example 3)

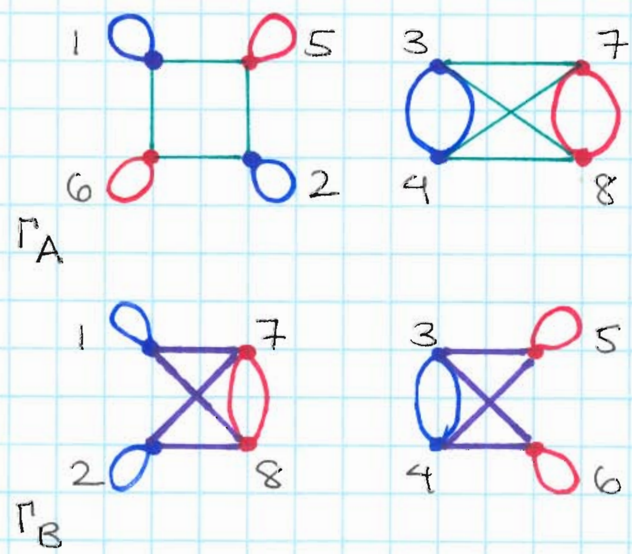

Figure 8.9 : third example of Lemma 8.1

9 Further questions

While exploring the preceding results, many other questions have come up that would be of interest to pursue. We provide some of these questions below:

- Is there a way to compute the number of possible edge sets that exist when given G_1 and G_2? (Excluding redundancy of isomorphisms)
- Do there exist more generalizations of examples we have referred to?
- Are there any consistent results when $G_1 = G_2$?
- Can you determine when Γ_A and Γ_B will be isomorphic?
- Are there any general results when G_1 has an odd number of vertices and is not regular?
- What happens when G_1 and G_2 both have an even number or vertices, but G_1 is not regular?
- What happens when G_1 is regular and has an odd number of vertices?
- Are there any general results when G_1 and G_2 have the same regularity versus different regularity?
- How often do symmetric Seidel pairs arise?

- What affects do the presence of loops in G_1 and/or G_2 have on Γ_A and Γ_B?

- Why is it that cospectral pairs of regular graphs are rarer than cospectral pairs of non-regular graphs?

A All 3-walks in Γ_A and Γ_B (Combinatorial Proof)

We provide a comprehensive list of all closed 3-walks W and Z in Γ_A and Γ_B respectively such that they are divided into the three types as described.

Type I: all closed 3-walks such that W is contained in G_1. There will be none containing vertices 4 or 5.

- $1 \to 2 \to (a)3 \to 1$
- $1 \to 2 \to (b)3 \to 1$
- $1 \to 3 \to (a)2 \to 1$
- $1 \to 3 \to (b)2 \to 1$
- $2 \to 1 \to 3 \to (a)2$
- $2 \to 1 \to 3 \to (b)2$
- $2 \to (a)3 \to 1 \to 2$
- $2 \to (b)3 \to 1 \to 2$
- $3 \to 1 \to 2 \to (a)3$
- $3 \to 1 \to 2 \to (b)3$

- $3 \to (a)2 \to 1 \to 3$
- $3 \to (b)2 \to 1 \to 3$

$|\text{Type I } W| = 12$

Type II: all closed 3-walks W in G_2. These will contain vertices 6, 7, and 8 or vertex 11, but not vertices 9 or 10.

- $6 \to 7 \to 8 \to 6$
- $6 \to 8 \to 7 \to 6$
- $7 \to 6 \to 8 \to 7$
- $7 \to 8 \to 6 \to 7$
- $8 \to 6 \to 7 \to 8$
- $8 \to 7 \to 6 \to 8$
- $11 \to (cw)11 \to (cw)11 \to (cw)11$
- $11 \to (cw)11 \to (cw)11 \to (ccw)11$
- $11 \to (cw)11 \to (ccw)11 \to (cw)11$
- $11 \to (cw)11 \to (ccw)11 \to (ccw)11$
- $11 \to (ccw)11 \to (cw)11 \to (cw)11$

- $11 \to (ccw)11 \to (cw)11 \to (ccw)11$
- $11 \to (ccw)11 \to (ccw)11 \to (cw)11$
- $11 \to (ccw)11 \to (ccw)11 \to (ccw)11$

$|\text{Type II } W| = 14$

Type III: W contains some edges in $\mathcal{E}$

Let us divide these up by the vertex at which the 3-walk begins and end.

Vertex 1:

- $1 \to 3 \to 7 \to 1$
- $1 \to 3 \to 8 \to 1$
- $1 \to 6 \to 7 \to 1$
- $1 \to 6 \to 8 \to 1$
- $1 \to 7 \to 3 \to 1$
- $1 \to 7 \to 6 \to 1$
- $1 \to 7 \to 8 \to 1$
- $1 \to 8 \to 3 \to 1$
- $1 \to 8 \to 6 \to 1$

- $1 \to 8 \to 7 \to 1$

Vertex 2:

- $2 \to (a)3 \to 9 \to 2$
- $2 \to (b)3 \to 9 \to 2$
- $2 \to 9 \to 3 \to (a)2$
- $2 \to 9 \to 3 \to (b)2$
- $2 \to 9 \to (a)10 \to 2$
- $2 \to 9 \to (b)10 \to 2$
- $2 \to 10 \to (a)9 \to 2$
- $2 \to 10 \to (b)9 \to 2$
- $2 \to 11 \to (cw)11 \to 2$
- $2 \to 11 \to (ccw)11 \to 2$

Vertex 3:

- $3 \to 1 \to 7 \to 3$
- $3 \to 1 \to 8 \to 3$

- $3 \to (a)2 \to 9 \to 3$
- $3 \to (b)2 \to 9 \to 3$
- $3 \to 7 \to 1 \to 3$
- $3 \to 7 \to 8 \to 3$
- $3 \to 8 \to 1 \to 3$
- $3 \to 8 \to 7 \to 3$
- $3 \to 9 \to 2 \to (a)3$
- $3 \to 9 \to 2 \to (b)3$

Vertex 4:

- $4 \to (a)5 \to 9 \to 4$
- $4 \to (a)5 \to 10 \to 4$
- $4 \to (b)5 \to 9 \to 4$
- $4 \to (b)5 \to 10 \to 4$
- $4 \to 9 \to 5 \to (a)4$
- $4 \to 9 \to 5 \to (b)4$

- $4 \rightarrow 9 \rightarrow (a)10 \rightarrow 4$
- $4 \rightarrow 9 \rightarrow (b)10 \rightarrow 4$
- $4 \rightarrow 10 \rightarrow 5 \rightarrow (a)4$
- $4 \rightarrow 10 \rightarrow 5 \rightarrow (b)4$
- $4 \rightarrow 10 \rightarrow (a)9 \rightarrow 4$
- $4 \rightarrow 10 \rightarrow (b)9 \rightarrow 4$

Vertex 5:

- $5 \rightarrow (a)4 \rightarrow 9 \rightarrow 5$
- $5 \rightarrow (a)4 \rightarrow 10 \rightarrow 5$
- $5 \rightarrow (b)4 \rightarrow 9 \rightarrow 5$
- $5 \rightarrow (b)4 \rightarrow 10 \rightarrow 5$
- $5 \rightarrow 9 \rightarrow 4(a) \rightarrow 5$
- $5 \rightarrow 9 \rightarrow 4 \rightarrow (b)5$
- $5 \rightarrow 9 \rightarrow (a)10 \rightarrow 5$
- $5 \rightarrow 9 \rightarrow (b)10 \rightarrow 5$

- $5 \rightarrow 10 \rightarrow 4 \rightarrow (a)5$
- $5 \rightarrow 10 \rightarrow 4 \rightarrow (b)5$
- $5 \rightarrow 10 \rightarrow (a)9 \rightarrow 5$
- $5 \rightarrow 10 \rightarrow (b)9 \rightarrow 5$
- $5 \rightarrow 11 \rightarrow (cw)11 \rightarrow 5$
- $5 \rightarrow 11 \rightarrow (ccw)11 \rightarrow 5$

Vertex 6:

- $6 \rightarrow 1 \rightarrow 7 \rightarrow 6$
- $6 \rightarrow 1 \rightarrow 8 \rightarrow 6$
- $6 \rightarrow 7 \rightarrow 1 \rightarrow 6$
- $6 \rightarrow 8 \rightarrow 1 \rightarrow 6$

Vertex 7:

- $7 \rightarrow 1 \rightarrow 3 \rightarrow 7$
- $7 \rightarrow 1 \rightarrow 6 \rightarrow 7$
- $7 \rightarrow 1 \rightarrow 8 \rightarrow 7$

- $7 \to 3 \to 1 \to 7$
- $7 \to 3 \to 8 \to 7$
- $7 \to 6 \to 1 \to 7$
- $7 \to 8 \to 1 \to 7$
- $7 \to 8 \to 3 \to 7$

Vertex 8:

- $8 \to 1 \to 3 \to 8$
- $8 \to 1 \to 6 \to 8$
- $8 \to 1 \to 7 \to 8$
- $8 \to 3 \to 1 \to 8$
- $8 \to 3 \to 7 \to 8$
- $8 \to 6 \to 1 \to 8$
- $8 \to 7 \to 1 \to 8$
- $8 \to 7 \to 3 \to 8$

Vertex 9:

- $9 \to 2 \to (a)3 \to 9$
- $9 \to 2 \to (b)3 \to 9$
- $9 \to 2 \to 10 \to (a)9$
- $9 \to 2 \to 10 \to (b)9$
- $9 \to 3 \to (a)2 \to 9$
- $9 \to 3 \to (b)2 \to 9$
- $9 \to 4 \to (a)5 \to 9$
- $9 \to 4 \to (b)5 \to 9$
- $9 \to 4 \to 10 \to (a)9$
- $9 \to 4 \to 10 \to (b)9$
- $9 \to 5 \to (a)4 \to 9$
- $9 \to 5 \to (b)4 \to 9$
- $9 \to 5 \to 10 \to (a)9$
- $9 \to 5 \to 10 \to (b)9$
- $9 \to (a)10 \to 2 \to 9$

- $9 \to (a)10 \to 4 \to 9$
- $9 \to (a)10 \to 5 \to 9$
- $9 \to (b)10 \to 2 \to 9$
- $9 \to (b)10 \to 4 \to 9$
- $9 \to (b)10 \to 5 \to 9$

Vertex 10:

- $10 \to 2 \to 9 \to (a)10$
- $10 \to 2 \to 9 \to (b)10$
- $10 \to 4 \to (a)5 \to 10$
- $10 \to 4 \to (b)5 \to 10$
- $10 \to 4 \to 9 \to (a)10$
- $10 \to 4 \to 9 \to (b)10$
- $10 \to 5 \to (a)4 \to 10$
- $10 \to 5 \to (b)4 \to 10$
- $10 \to 5 \to 9 \to (a)10$

- $10 \to 5 \to 9 \to (b)10$
- $10 \to (a)9 \to 2 \to 10$
- $10 \to (a)9 \to 4 \to 10$
- $10 \to (a)9 \to 5 \to 10$
- $10 \to (b)9 \to 2 \to 10$
- $10 \to (b)9 \to 4 \to 10$
- $10 \to (b)9 \to 5 \to 10$

Vertex 11:

- $11 \to 2 \to 11 \to (cw)11$
- $11 \to 2 \to 11 \to (ccw)11$
- $11 \to 5 \to 11 \to (cw)11$
- $11 \to 5 \to 11 \to (ccw)11$
- $11 \to (cw)11 \to 2 \to 11$
- $11 \to (cw)11 \to 5 \to 11$
- $11 \to (cw)11 \to (cw)11 \to (cw)11$

- $11 \rightarrow (cw)11 \rightarrow (cw)11 \rightarrow (ccw)11$
- $11 \rightarrow (cw)11 \rightarrow (ccw)11 \rightarrow (cw)11$
- $11 \rightarrow (cw)11 \rightarrow (ccw)11 \rightarrow (ccw)11$
- $11 \rightarrow (ccw)11 \rightarrow 2 \rightarrow 11$
- $11 \rightarrow (ccw)11 \rightarrow 5 \rightarrow 11$
- $11 \rightarrow (ccw)11 \rightarrow (cw)11 \rightarrow (cw)11$
- $11 \rightarrow (ccw)11 \rightarrow (cw)11 \rightarrow (ccw)11$
- $11 \rightarrow (ccw)11 \rightarrow (ccw)11 \rightarrow (cw)11$
- $11 \rightarrow (ccw)11 \rightarrow (ccw)11 \rightarrow (ccw)11$

$|\text{Type III } W| = 128$

Type I′ for 3-walks Z contained in G_1 and Type II′ 3-walks Z in G_2 will be the same as Type I and II W walks respectively. Thus, we now list all of the Type III′ 3-walks Z.

Type III′: Z is a 3-walk containing some edges in $\mathcal{E}^{\mathcal{C}}$.

Vertex 1:

- $1 \rightarrow 3 \rightarrow 10 \rightarrow 1$

- $1 \to 3 \to 11 \to 1$
- $1 \to 9 \to (a)10 \to 1$
- $1 \to 9 \to (b)10 \to 1$
- $1 \to 10 \to 3 \to 1$
- $1 \to 10 \to (a)9 \to 1$
- $1 \to 10 \to (b)9 \to 1$
- $1 \to 11 \to 3 \to 1$
- $1 \to 11 \to (cw)11 \to 1$
- $1 \to 11 \to (ccw)11 \to 1$

Vertex 2:

- $2 \to (a)3 \to 6 \to 2$
- $2 \to (b)3 \to 6 \to 2$
- $2 \to 6 \to 3 \to (a)2$
- $2 \to 6 \to 3 \to (b)2$
- $2 \to 6 \to 7 \to 2$

- $2 \to 6 \to 8 \to 2$
- $2 \to 7 \to 6 \to 2$
- $2 \to 7 \to 8 \to 2$
- $2 \to 8 \to 6 \to 2$
- $2 \to 8 \to 7 \to 2$

Vertex 3:

- $3 \to 1 \to 10 \to 3$
- $3 \to 1 \to 11 \to 3$
- $3 \to (a)2 \to 6 \to 3$
- $3 \to (b)2 \to 6 \to 3$
- $3 \to 6 \to 2 \to (a)3$
- $3 \to 6 \to 2 \to (b)3$
- $3 \to 10 \to 1 \to 3$
- $3 \to 11 \to 1 \to 3$
- $3 \to 11 \to (cw)11 \to 3$

- $3 \to 11 \to (ccw)11 \to 3$

Vertex 4:

- $4 \to (a)5 \to 6 \to 4$
- $4 \to (a)5 \to 7 \to 4$
- $4 \to (b)5 \to 6 \to 4$
- $4 \to (b)5 \to 7 \to 4$
- $4 \to 6 \to 5 \to (a)4$
- $4 \to 6 \to 5 \to (b)4$
- $4 \to 6 \to 7 \to 4$
- $4 \to 7 \to 5 \to (a)4$
- $4 \to 7 \to 5 \to (b)4$
- $4 \to 7 \to 6 \to 4$
- $4 \to 11 \to (cw)11 \to 4$
- $4 \to 11 \to (ccw)11 \to 4$

Vertex 5:

- $5 \to (a)4 \to 6 \to 5$
- $5 \to (a)4 \to 7 \to 5$
- $5 \to (b)4 \to 6 \to 5$
- $5 \to (b)4 \to 7 \to 5$
- $5 \to 6 \to 4 \to (a)5$
- $5 \to 6 \to 4 \to (b)5$
- $5 \to 6 \to 7 \to 5$
- $5 \to 6 \to 8 \to 5$
- $5 \to 7 \to 4 \to (a)5$
- $5 \to 7 \to 4 \to (b)5$
- $5 \to 7 \to 6 \to 5$
- $5 \to 7 \to 8 \to 5$
- $5 \to 8 \to 6 \to (a)5$
- $5 \to 8 \to 6 \to (b)5$
- $5 \to 8 \to 7 \to (a)5$

- $5 \to 8 \to 7 \to (b)5$

Vertex 6:

- $6 \to 2 \to (a)3 \to 6$
- $6 \to 2 \to (b)3 \to 6$
- $6 \to 2 \to 7 \to 6$
- $6 \to 2 \to 8 \to 6$
- $6 \to 3 \to (a)2 \to 6$
- $6 \to 3 \to (b)2 \to 6$
- $6 \to 4 \to (a)5 \to 6$
- $6 \to 4 \to (b)5 \to 6$
- $6 \to 4 \to 7 \to 6$
- $6 \to 5 \to (a)4 \to 6$
- $6 \to 5 \to (b)4 \to 6$
- $6 \to 5 \to 7 \to 6$
- $6 \to 5 \to 8 \to 6$

- $6 \to 7 \to 2 \to 6$
- $6 \to 7 \to 4 \to 6$
- $6 \to 7 \to 5 \to 6$
- $6 \to 8 \to 2 \to 6$
- $6 \to 8 \to 5 \to 6$

Vertex 7:

- $7 \to 2 \to 6 \to 7$
- $7 \to 2 \to 8 \to 7$
- $7 \to 4 \to (a)5 \to 7$
- $7 \to 4 \to (b)5 \to 7$
- $7 \to 4 \to 6 \to 7$
- $7 \to 5 \to (a)4 \to 7$
- $7 \to 5 \to (b)4 \to 7$
- $7 \to 5 \to 6 \to 7$
- $7 \to 5 \to 8 \to 7$

- $7 \to 6 \to 2 \to 7$
- $7 \to 6 \to 4 \to 7$
- $7 \to 6 \to 5 \to 7$
- $7 \to 8 \to 2 \to 7$
- $7 \to 8 \to 5 \to 7$

Vertex 8:

- $8 \to 2 \to 6 \to 8$
- $8 \to 2 \to 7 \to 8$
- $8 \to 5 \to 6 \to 8$
- $8 \to 5 \to 7 \to 8$
- $8 \to 6 \to 2 \to 8$
- $8 \to 6 \to 5 \to 8$
- $8 \to 7 \to 2 \to 8$
- $8 \to 7 \to 5 \to 8$

Vertex 9:

- $9 \to 1 \to 10 \to (a)9$
- $9 \to 1 \to 10 \to (b)9$
- $9 \to (a)10 \to 1 \to 9$
- $9 \to (b)10 \to 1 \to 9$

Vertex 10:

- $10 \to 1 \to 3 \to 10$
- $10 \to 3 \to 1 \to 10$
- $10 \to (a)9 \to 1 \to 10$
- $10 \to (b)9 \to 1 \to 10$

Vertex 11:

- $11 \to 1 \to 3 \to 11$
- $11 \to 1 \to 11 \to (cw)11$
- $11 \to 1 \to 11 \to (ccw)11$
- $11 \to 3 \to 1 \to 11$
- $11 \to 3 \to 11 \to (cw)11$

- $11 \to 3 \to 11 \to (ccw)11$
- $11 \to 4 \to 11 \to (cw)11$
- $11 \to 4 \to 11 \to (ccw)11$
- $11 \to (cw)11 \to 1 \to 11$
- $11 \to (cw)11 \to 3 \to 11$
- $11 \to (cw)11 \to 4 \to 11$
- $11 \to (cw)11 \to (cw)11 \to (cw)11$
- $11 \to (cw)11 \to (cw)11 \to (ccw)11$
- $11 \to (cw)11 \to (ccw)11 \to (cw)11$
- $11 \to (cw)11 \to (ccw)11 \to (ccw)11$
- $11 \to (ccw)11 \to 1 \to 11$
- $11 \to (ccw)11 \to 3 \to 11$
- $11 \to (ccw)11 \to 4 \to 11$
- $11 \to (ccw)11 \to (cw)11 \to (cw)11$
- $11 \to (ccw)11 \to (cw)11 \to (ccw)11$

- $11 \to (ccw)11 \to (ccw)11 \to (cw)11$

- $11 \to (ccw)11 \to (ccw)11 \to (ccw)11$

$|\text{Type III}'\ Z| = 128$

B Symmetric matrices have real eigenvalues

Our ultimate goal is to present the proof of the Fundamental Theorem of Real Symmetric Matrices. In order to do so, we must first present and provide the proofs of two other theorems that will be needed in this proof.

B.1 Schur's Lemma

We begin with the theorem known as Schur's Lemma. This will be used in the proof of the next preparatory theorem whose proof we will be working through.

Theorem B.1. *Let A be an $n \times n$ (complex) matrix. There is a unitary matrix U such that $U^{-1}AU$ is upper triangular.*

Proof. We will prove Schur's Lemma by induction.

If $n = 1$, then Schur's Lemma is trivially true.

Let us assume this theorem holds for all $m \times m$ matrices such that $1 \leq m \leq$ (n−1).

We will now show that it holds for an $n \times n$ matrix A.

Let λ_1 be an eigenvalue of A, and let $\mathbf{v}_1$ be a corresponding unit eigenvector.

We know that A, and every complex matrix, will have at least 1 eigenvalue by the Fundamental Theorem of Algebra (Section 9.1 of [4]).

We now expand $\mathbf{v}_1$ so that it becomes a basis for $\mathbb{C}^n$ by choosing $\mathbf{v}_2$,...,$\mathbf{v}_n$ so $\mathbf{v}_1$,...,$\mathbf{v}_n$ is an orthonormal basis.

Using the Gram-Schmidt process we can then transform it into an orthonormal basis

$$\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n.$$

Let U_1 be the unitary matrix with j^{th} column vector $\mathbf{v}_j$. The first column vector of AU_1 will be $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$.

We know the i^{th} row vector of U_1^* is $\mathbf{v}_i^*$. Since the vectors $\mathbf{v}_j$ are orthogonal, $\mathbf{v}_i^*\mathbf{v}_j = 0$ for $i \neq j$.

So the first column vector of $U_1^*AU_1$ will be

$$U_1^*(\lambda_1\mathbf{v}_1) = \begin{pmatrix} \lambda_1 \\ 0 \\ : \\ : \\ 0 \end{pmatrix}.$$

Using this, we can write

$$U_1^* A U_1 = \begin{pmatrix} \lambda_1 & * & * & .. & .. & * \\ 0 & & & & & \\ 0 & & & & & \\ : & & & A_1 & & \\ : & & & & & \\ 0 & & & & & \end{pmatrix}, \tag{B.1}$$

where A_1 is an $(n-1)\times(n-1)$ submatrix.

Our induction hypothesis indicates there is an $(n-1)\times(n-1)$ unitary matrix C such that $C^* A_1 C = B$ for an upper triangular matrix B.

Let

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & .. & .. & 0 \\ 0 & & & & & \\ 0 & & & & & \\ : & & & C & & \\ : & & & & & \\ 0 & & & & & \end{pmatrix}. \tag{B.2}$$

Since C is unitary, we can show that U_2 is also a unitary matrix.

$$\begin{aligned} U_2^* U_2 &= \begin{pmatrix} 1 & 0 & 0 & .. & .. & 0 \\ 0 & & & & & \\ 0 & & & & & \\ : & & & C^* & & \\ : & & & & & \\ 0 & & & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & .. & 0 \\ 0 & & & & & \\ 0 & & & & & \\ : & & & C & & \\ : & & & & & \\ 0 & & & & & \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & .. & .. & 0 \\ 0 & & & & & \\ 0 & & & & & \\ : & & & C^*C & & \\ : & & & & & \\ 0 & & & & & \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 & .. & .. & 0 \\ 0 & & & & & \\ 0 & & & & & \\ : & & & I & & \\ : & & & & & \\ 0 & & & & & \end{pmatrix} = I$$

and thus U_2 is unitary.

We now consider $U = U_1 U_2$ and calculate

$$U^* U = U_2^* (U_1^* U_1) U_2 = U_2^* I U_2 = U_2^* U_2 = I.$$

This indicates that U is also unitary.

Using

$$U^* A U = U_2^* U_1^* A U_1 U_2 \tag{B.3}$$

we can combine (2.3) with (2.1) and (2.2):

$$U^* A U =$$

$$\begin{pmatrix} 1 & 0 & .. & .. & 0 \\ 0 & & & & \\ : & & C^* & & \\ : & & & & \\ 0 & & & & \end{pmatrix} \begin{pmatrix} \lambda_1 & * & .. & .. & * \\ 0 & & & & \\ : & & A_1 & & \\ : & & & & \\ 0 & & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & .. & .. & 0 \\ 0 & & & & \\ : & & C & & \\ : & & & & \\ 0 & & & & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & * & .. & .. & * \\ 0 & & & & \\ : & & C^* A_1 & & \\ : & & & & \\ 0 & & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & .. & .. & 0 \\ 0 & & & & \\ : & & C & & \\ : & & & & \\ 0 & & & & \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & * & .. & .. & * \\ 0 & & & & \\ : & & C^*A_1C & & \\ : & & & & \\ 0 & & & & \end{pmatrix}$$

We obtain

$$U^*AU = \begin{pmatrix} \lambda_1 & * & .. & .. & * \\ 0 & & & & \\ : & & B & & \\ : & & & & \\ 0 & & & & \end{pmatrix}.$$

Since B is upper triangular, this tells us U^*AU is upper triangular as well. □

Now that we have completed this proof, we can use Schur's Lemma in the proof of the Spectral Theorem for Hermitian Matrices. This next theorem will then be used in the proof of the Fundamental Theorem of Real Symmetric Matrices.

B.2 Spectral Theorem for Hermitian Matrices

Theorem B.2. *If A is a Hermitian matrix, there exists a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix. Also, all eigenvalues of A are real.*

Proof. We will begin by proving the first part of the theorem.

From Schur's Lemma, we have that there is a unitary matrix U such that $U^{-1}AU$ is upper triangular.

Given that U is unitary, we see that $U^{-1} = U^*$ and $U^*U = I$. Because A is Hermitian, $A^* = A$.

So,

$$(U^{-1}AU)^* = (U^*AU)^* = U^*A^*(U^*)^* = U^*AU = U^{-1}AU.$$

Therefore, $U^{-1}AU$ is Hermitian as well.

The conjugate transpose of an upper triangular matrix will be a lower triangular matrix. Knowing that $U^{-1}AU$ is Hermitian indicates it is both upper triangular and lower triangular, so it must be a diagonal matrix D: $U^{-1}AU = D$.

Therefore, we now know that the matrix A is unitarily diagonalizable, $U^{-1}AU = D$.

We can now show that the eigenvalues of A are real. Since D is a diagonal matrix, we know that the entries along it's diagonal will be the eigenvalues of A.

We also know that D is Hermitian, so $D = D^*$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & . & . & 0 \\ & \lambda_2 & & & . & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & \lambda_n \end{pmatrix}$$

$$D^* = \begin{pmatrix} \lambda_1^* & 0 & 0 & . & . & 0 \\ & \lambda_2^* & & & . & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & \lambda_n^* \end{pmatrix}$$

Since $D = D^*$, $\lambda_j = \lambda_j^*$ for $j = 1, ..., n$.

Therefore, the eigenvalues $\lambda_1, ..., \lambda_n$ must be real. □

We are now able to use this important conclusion in the proof of the Fundamental Theorem of Real Symmetric Matrices.

B.3 Fundamental Theorem of Real Symmetric Matrices

Theorem B.3. *Every $n \times n$ real symmetric matrix has n real eigenvalues (counted with algebraic multiplicity) and is diagonalizable by a real orthogonal matrix.*

Proof. We know every real $n \times n$ symmetric matrix A is also Hermitian.

By the Spectral Theorem for Hermitian Matrices, A is diagonalizable with real eigenvalues $\lambda_1, ..., \lambda_n$.

Therefore, A has n real eigenvalues counted with algebraic multiplicity.

We must count the eigenvalues with multiplicity as a result of the following theorem found on p.313 of [4].

Theorem B.4. ***A Criterion for Diagonalization:*** *An $n \times n$ matrix A is diagonalizable if and only if the algebraic multiplicity of each (possibly complex) eigenvalue is equal to its geometric multiplicity.*

From the Spectral Theorem for Hermitian Matrices we also know that A is unitarily diagonalizable. We will let it be diagonalized by the unitary matrix U.

By Theorem 5.2 of [4], **Matrix Summary of Eigenvalues of A**, and its proof (p.306), the column vectors of U will be eigenvectors of A. We can determine these eigenvectors by row-reducing $A - \lambda_i I$ for $i = 1, ..., n$.

We have already proved the λ_i are real. We will now row-reduce over the real numbers.

This tells us the row-reduced echelon form of $A - \lambda_i I$ will have only real elements.

This will be best illustrated through an example.

Example: Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

We will first determine the eigenvalues of A and their corresponding row-reduced echelon forms of $A - \lambda_i I$.

The characteristic polynomial of A is:

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}$$
$$= (2-\lambda)[(1-\lambda)^2 - 1]$$

$$= -\lambda(\lambda - 2)^2$$

$$\lambda_1 = 0, \lambda_2 = \lambda_3 = 2$$

Now that we have found the eigenvalues of A, we can determine the row-reduced echelon forms of the $A - \lambda_i I$ and show all their entries are real numbers.

$$\lambda_1 = 0$$

$$A - \lambda_1 I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_2 = \lambda_3 = 2$$

$$A - 2 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus showing the row-reduced echelon forms of $A - \lambda_i I$ contain only real entries.

Since A is diagonalizable, $A - \lambda_i I$ in row-reduced echelon form will have

$$\dim(\text{nullspace}) = \text{geometric multiplicity of } \lambda_i$$

$$\text{geometric multiplicity of } \lambda_i = \text{algebraic multiplicity of } \lambda_i.$$

If, for $M = A - \lambda_i I$, M is an $n \times n$ real matrix with rank $M = k \leq n$, then the nullspace of M is a subspace of $\mathbb{R}^n$ of dimension $n - k$. Therefore the nullspace of M has a basis consisting of vectors in $\mathbb{R}^n$.

The Gram-Schmidt process lets us assume the basis of each eigenspace is orthonormal.

If we let vectors of these orthonormal bases of the eigenspaces form the columns of the matrices, $C_1, C_2, ..., C_r$, we obtain a real orthogonal matrix C that diagonalizes A.

$$C = \begin{pmatrix} C_1 & & & & & \\ & C_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & C_r \end{pmatrix}$$

□

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