THE PERMUTATIONS OF PERIODIC POINTS IN QUADRATIC POLYNOMIALS

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Part 1. Preface

I began my independent research with the following question:

Question Let us take $f(z) = z^2 + c$ to be a function in the complex plane. If we let c travel on a simple, closed, and continuous loop, what is the movement of the period five points of the quadratic?

To explore this question, we need to start from the most general and work our way to more specific. Part II is a summary of a complex analysis course. We begin with an introduction to the complex plane, e^z and polynomials. We will prove the Fundamental Theorem of Algebra using both Liouville's Theorem and Rouché's Theorem.

In Part 2, we will first take a brief look at the topology of loops. We will take into account the notion of homotopy, which will give us equivalence classes of loops. We will also look at the algebraic function $g(z) = z - \sqrt{c}$ since we can then make connection to our function. We will introduce the ideas of the Riemann Surface and ramification points as it applies to our situation.

In Parts 3 and 4 we delve into our problem. We will bring together the ideas of iteration and periodic points. Then using the concepts of the homotopy class of loops we create loops in the c-plane that induce a permutation of the periodic points of $f(z)$ in the z-plane. For each c-value in a polynomial of degree m, we will have m roots, when counted according to multiplicity, by the Fundamental Theorem of Algebra. As our c travels, so do our periodic points. Our intuition tells us that if we let c travel in a simple, closed and continuous loop in the c-plane, our periodic points will also travel in some form of a closed loop in the z-plane. Our question then becomes whether or not we can predict the movement of our periodic points from the movement of c. We will closely examine the permutations of fixed points and period two points. We will then generally discuss the permutations of period three points, and only briefly touch on the permutations of period four points.

We conclude this paper with some thoughts about what might occur for further iterates of our quadratic and other topics to explore.

2. Introduction to Complex Numbers

The complex plane, denoted $\mathbb{C}$, first made its appearance through the work of Girolamo Cardano. (cf., [10], page 1). Its usefulness comes from being able to evaluate the even root of a negative number and the complex plane has the *imaginary unit* i with the property $i^2 = -1$ (cf., [1], page 1). Complex numbers are in the form $z = a + bi$, where $a, b \in \Re$. If $b = 0$, z is a real number. If $a = 0$, z is then *purely imaginary*. We say that a is the real part of the expression, and b is the imaginary part of the expression c (cf., [7], pages 4-5).

We can now consider numbers as being in the complex plane. Let us consider the complex numbers as ordered pairs of real numbers, that is, (a, b). Since a is the real part of z, we will consider the x-axis as the *real axis*. Similarly, since b is the imaginary part of a complex number, we can take the y-axis as the *imaginary axis* (cf., [4], page 4). For example, say we wanted to plot the complex vector $2 + 3i$. We would then have the point $(2, 3)$.

Let us take a point (x, y) in the real plane. We recall that, using the distance formula, we find the distance from the point to the origin by taking $\sqrt{x^2 + y^2}$. In the complex plane, we can then find the length of the vector z, the *modulus* of z, as $\sqrt{a^2 + b^2}$, denoted $|z|$. We can use the modulus of z to determine a circle in the complex plane that is centered at the origin with the radius equal to $|z|$ (cf., [4], page 5).

3. Using e^z in the Complex Plane

In order to analyze the geometry of the permutations of periodic points, we want to look at closed loops in the complex plane. Circles are easy loops to create in the complex plane using the transcendental number e. We know from calculus that for any real $x \in \Re$, we can define e^x by its Maclaurin series as

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots.$$

We can extend our definition of the exponential function from the real plane to the complex plane. We then have that $x = i\theta$, and

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \dots.$$

Rearranging the series shows that

$$e^{i\theta} = \left(1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - \dots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} - \dots\right).$$

Since

$$\cos\theta = \left(1 - \frac{(\theta)^2}{2!} + \frac{(\theta)^4}{4!} - \dots\right)$$

and

$$\sin\theta = \left(\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} - \dots\right),$$

we can then have

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(cf., [7], page 25)}$$

We are going to use t as a real parameter that represents the number of turns about the origin. For example, for $\theta = \frac{\pi}{2}$ we take $t = \frac{1}{4}$. We can now let $\theta = 2\pi t$, and have

$$e^{2\pi it} = \cos 2\pi t + i\sin 2\pi t$$

Thus, we have a way of considering e, i and π in one setting. Some key values for $e^{2\pi it}$ are;

$$\begin{aligned} e^{2\pi i(0)} &= 1 \\ e^{2\pi i\frac{1}{4}} &= i \\ e^{2\pi i\frac{1}{2}} &= -1 \\ e^{2\pi i\frac{3}{4}} &= -i \\ e^{2\pi i} &= 1. \end{aligned}$$

As we let t increase from 0 to 1, we notice that the parametric curve $\sigma(t) = e^{2\pi it}$ traverses the points on the unit circle. Hence, to traverse a circle in the counterclockwise direction with a fixed center at any complex number, v, and fixed radius length, $r \in \Re$, we use the parametric equation

$$L(t) = v + r(e^{2\pi it}).$$

4. Polynomials in the Complex Plane

With this short introduction to the complex numbers, we can begin working with polynomials over the complex plane. For the remainder of this paper, we will be analyzing polynomials whose coefficients and variables take values in the complex plane, in particular we will consider the zeros of polynomials.

Definition Let $Q(z)$ be a polynomial in the complex plane and k a positive integer. The complex number a is a zero of a polynomial, $P(z)$, if $P(z) = (z-a)^k Q(z)$ with $P(a) = 0$ and $Q(a) \neq 0$. In this case, we call a a zero of *multiplicity* k (cf., [4], page 60).

In order to examine the zeros of a polynomial, we first examine the conditions under which a zero will exist. More precisely, we want to consider whether or not a polynomial will always have a zero in the complex plane. By the Fundamental Theorem of Algebra any polynomial must have a zero in $\mathbb{C}$. Let us first look at a definition and at Liouville's Theorem before we examine the Fundamental Theorem of Algebra.

Definition A function is said to be *entire* if it is differentiable everywhere in the complex plane (cf., [4], page 35).

Theorem 1. *[Liouville's Theorem] A bounded entire function is constant (cf., [4], page 59).*

Theorem 2. *[Fundamental Theorem of Algebra] Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$ (cf., [4], page 59).*

Proof. Suppose that we have a polynomial $P(z)$ such that

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$$

with $a_i \in \mathbb{C}$ and $P(z)$ is nonconstant and does not equal zero for all $z \in \mathbb{C}$. We can then take

$$f(z) = \frac{1}{P(z)}$$

as an entire function on the complex plane. Since $P(z)$ is nonconstant, $|P(z)| \to \infty$ as $z \to \infty$. Hence, $f(z)$ is bounded, and by Liouville, is thus constant. Yet in order for $f(z)$ to be constant, $P(z)$ would have to be constant, which contradicts our assumption. Therefore, every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$. $\square$

Hence, for complex polynomials we know that we will have at least one zero in the complex plane. Let us now use Rouché's Theorem to prove that a polynomial, of degree m, will have a maximum of m zeros, when counted according to multiplicity.

Definition We say that f is *analytic* inside and on a regular closed curve if f is differentiable there (cf., [4], page 35).

Theorem 3. *[Rouché's Theorem] Suppose that f and g are analytic inside and on a regular closed curve γ and*

$$|f(z) - g(z)| < |f(z)|$$

for all $z \in \gamma$. Then f and g have the same number of zeros inside γ (cf., [4], page 113; [10], page 232).

An Alternative Proof of the Fundamental Theorem of Algebra. Let

$$\begin{aligned} f(z) &= z^m, \\ g(z) &= z^m + a_1 z^{m-1} + ... + a_m. \end{aligned}$$

Since $f(z)$ has m zeros when counted according to multiplicity, our goal in this proof is to show that $g(z)$ has the same number of zeros as $f(z)$. In order to prove that $g(z)$ has m zeros when counted according to multiplicity, we want to show that $|f(z) - g(z)| < |f(z)|$ for all $z \in \mathbb{C}$. We are given that,

$$\begin{aligned} |f(z) - g(z)| &= \left| z^m - z^m + a_1 z^{m-1} + \cdots + a_m \right| \\ &= \left| a_1 z^{m-1} + \cdots + a_m \right| \\ &= |z^m| \left| \frac{a_1}{z} + \cdots + \frac{a_m}{z^m} \right|. \end{aligned}$$

It follows that

$$\left| \frac{1}{z^m} \right| |f(z) - g(z)| = \left| \frac{a_1}{z} + \cdots + \frac{a_m}{z^m} \right|. \tag{1}$$

Hence, as $z \to \infty$, $\left| \frac{a_1}{z} + \cdots + \frac{a_m}{z^m} \right| \to 0$.

Earlier in this section, we defined a circle in the complex plane as having a radius, say R_0, equal to the modulus of z. Let us consider another circle with a very large radius, R, centered at the origin. Let $R_0 > R$, so $|z| > R$. Then

$$\left| \frac{a_1}{z} + ... + \frac{a_m}{z^m} \right| < 1. \tag{2}$$

Using equations 1 and 2, we have,

$$\begin{aligned} \left| \frac{1}{z^m} \right| |f(z) - g(z)| &< 1 \text{ for } z \in \mathbf{C} \\ |f(z) - g(z)| &< |z^m| = |f(z)|. \end{aligned}$$

Hence, by Rouché, $g(z)$ has the same number of zeros as $f(z)$, that is, m zeros according to multiplicity. Therefore, any polynomial of degree m over $\mathbb{C}$ has m zeros when counted according to multiplicity. □

Part 2. Background of Riemann Surfaces and the Case of $z = \sqrt{c}$

5. Loops

Throughout this paper, we will be looking at simple, continuous and closed loops orientated in the counter-clockwise direction. Let us look at loops in an annulus.

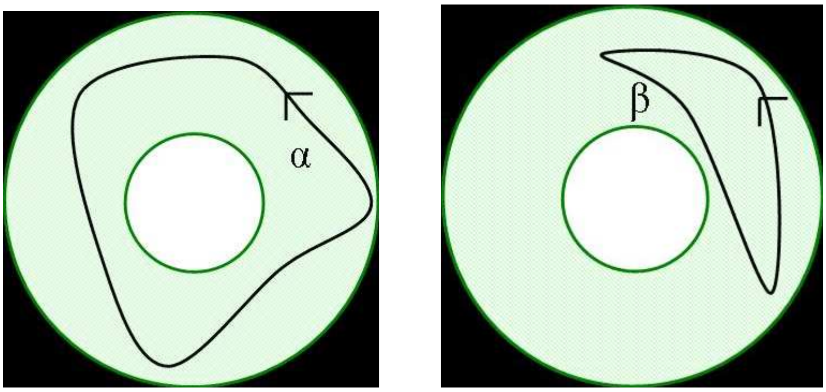

Figure 1. Two Loops in an Annulus

In the annulus, let us create a loop α that contains the hole of the annulus. Suppose we want to continuously shrink this loop into a single point. We notice that in order to do so, we would have to somehow cross the hole. Thus, in an annulus, we cannot shrink α into a single point when α contains the hole.

In contrast, let us take a loop β in the annulus that does not contain the hole. If we now try to continually shrink β into a single point, we can. We then have two different types of loops: those that can shrink continuously into a point and those that cannot. For example, if we generate another loop γ that contains the hole in the annulus, we realize that it too cannot continuously shrink into a single point. For us, γ is just as good as α since γ can be continuously deformed into α (cf., [3], pages 19-21).

Let us now work on a topological space, X, instead of an annulus. We can still deform loops into other loops. Deformations give rise to an equivalence relation on loops called *homotopy*: Loops α and γ are homotopic if γ can be continuously deformed into α. Based on continuous deformation, we have equivalence classes, that is *homotopy classes*. For example, β is not in the same class as γ and α. To get a clear idea of what is going on, we pick a starting point, that is a base point, at $p \in X$. We can define a variety of closed loops oriented in the counter-clockwise direction that can be deformed into other simple closed loops all oriented in the counter-clockwise direction that start and end at our base point. Let us represent our holes as $c_1^*, c_2^*, \ldots, c_j^*, \ldots$. Then $X = \mathbb{C} - \{c_1^*, c_2^*, \ldots, c_j^*, \ldots\}$. We will denote a loop that contains c_1^* as γ_1, etc. The homotopy class of a loop γ_i is then $\langle\gamma_i\rangle$.

We can now consider multiplication of loops. Multiplication of loops occurs when we choose a base point, $p \in X$,and traverse first a representative of $\langle\gamma_j\rangle$ and then follow it by a representative of $\langle\gamma_i\rangle$. The class of all loops formed by first traversing a γ_j and then a γ_i is represented by

$$\langle\gamma_j\rangle \cdot \langle\gamma_i\rangle = \langle\gamma_j \cdot \gamma_i\rangle.$$

We are assuming this operation on homotopy classes is well-defined and gives a group structure on the equivalence class of loops. We know that multiplication is associative. That is,

$$\langle\gamma_j \cdot \gamma_i\rangle \cdot \langle\gamma_s\rangle = \langle\gamma_j\rangle \cdot \langle\gamma_i \cdot \gamma_s\rangle.$$

We also have an identity element, which we will denote as $\langle\gamma_0\rangle$. For us, our identity element is the homotopy class of curves that do not contain any holes, since these curves are all equivalent to the base point.

Lastly, each element of our group has an inverse. We are taking loops in the counter-clockwise direction. The inverse of the homotopy class $\langle\gamma_j\rangle$, which we denote as $\langle\gamma_j^{-1}\rangle$, is our loop $\langle\gamma_j\rangle$ taken in the clockwise direction,i.e. γ_j^{-1} is γ_j taken clockwise.

Our group is called the *the fundamental group of X based at p*, and denoted by $\pi_1(X, p)$. The elements of our group are then generated by the classes

$$\langle\gamma_0\rangle, \langle\gamma_1\rangle \ldots \langle\gamma_j\rangle \ldots$$

and all elements are the products of the generators (cf., [3], pages 87-95).

6. Background of Riemann Surfaces and the Case of $z = \sqrt{c}$

In this part we will be considering the zeros of the complex polynomial $z^2 - c = 0$. We notice that for $f(z) = 0$, for each value of c we have two values for z. For example, for $c = 4, z = \pm 2$.

Using polar coordinates, we can rewrite c as $re^{2\pi it}$, for a fixed r and t. If we increase t from 0 to 1, c traverses a simple, closed loop around the origin that begins and ends at the real number r in the c-plane. Let us choose $r \neq 0$ we will look at a simple case first, and then generalize it to explain all cases. Let $r = 1$, so c is traversing the unit circle. At each value of c in the c-plane, we can then solve $f(z) = 0$ for z and plot these z values in the z-plane. We have that

$$\begin{aligned} z &= \sqrt{re^{2\pi it}} \\ z &= e^{\pi it}\sqrt{r} \end{aligned}$$

It is helpful to view the mapping from the c-plane to the z-plane in both a table and a figure. See Table 1 and Figure 2.

t	$c = e^{2\pi it}$	$z = e^{\pi it}\sqrt{1}$
0	1	1
1/2	-1	i
1	1	-1

Table 1. Mapping

From Table 1 and Figure 2, we notice that as c makes one complete turn about the origin in the c-plane, we have only completed half a turn about the origin in the z-plane. In essence, we ended in the z-plane at the negative value of our starting value. The question we want to consider is whether or not for any c that is a simple closed loop the starting value of z will differ from the ending z-value. Let $z = \sqrt{r}e^{\pi it}$. At $t = 0$ we have $z = \sqrt{r}$. Now let $t = 1$. At $t = 1$ we are at the same place in the c-plane as we were when $t = 0$. Yet in the z-plane we have that $z = \sqrt{r}e^{\pi i} = -\sqrt{r}$.

Let us now try constructing the Riemann Surface, which is an abstract manner of viewing what is occurring. In order to construct the surface we want to take two replicas of the c-plane each called a sheet. We call one sheet I and the other sheet II. Since our function is multi-valued, let I contain the positive values of z, and II contain the negative values of z. So I contains the points $(c, \sqrt{c})$, and II contains the points $(c, -\sqrt{c})$. Now suppose that we cut each sheet on the positive real axis.

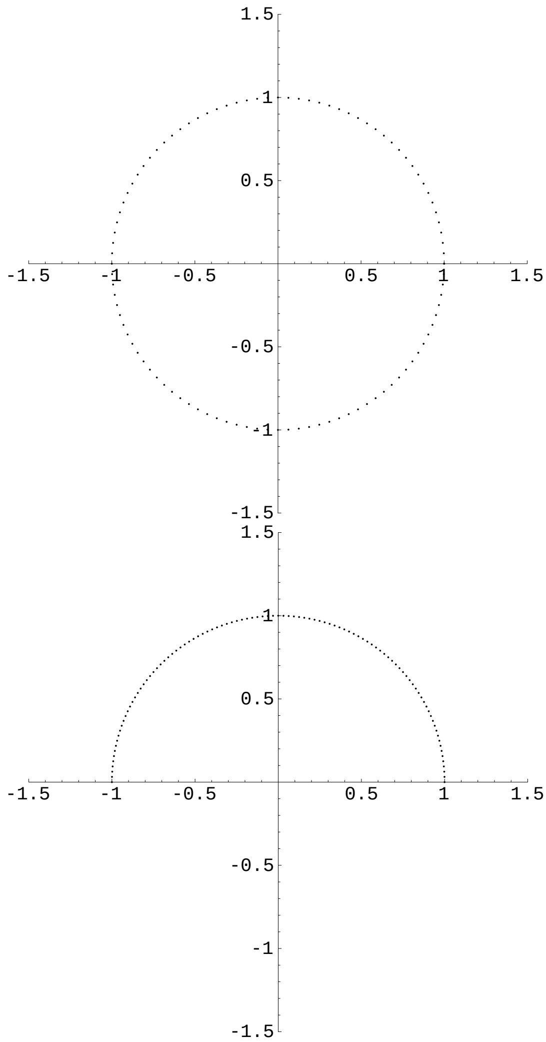

FIGURE 2. $c = e^{2\pi it}, z = e^{\pi it}\sqrt{1}$

We can think of this cut as breaking the bond that links the first and fourth quadrants together. Let us label the edge of the first quadrant by the cut with a + sign, and the other side of the cut in the fourth quadrant with a - sign on both sheets. In total we should have two + markings and two - markings.

Since both sheets are the complex plane, we can now perform some old-fashioned pasting. We want a + and a - to be pasted together, so let us take a + from I and paste it to a - from II, and vice-versa. We have now created a way to move from I to II. For example, previously, we saw that as c travelled along a circle centered at the origin, we started at the value $z = \sqrt{r_0}$, in I, and ended at the value $z = -\sqrt{r_0}$, which is in II. Thus, we travelled from one sheet to the other.

When will we be able to pass from one sheet to the next? In order to travel between sheets, we need to land on a point/s that is common in both sheets, that is we want our points $(c, \sqrt{c})$ and $(c, -\sqrt{c})$ to be the same. For our function $z = \sqrt{c}$, that will occur when $c = 0$, or ∞. These values for c control our path. For instance, if we travel in a loop around the particular c-value, we arrive on the common point in the sheets and are thus able to travel between them. If we do not, then we remain on the sheet where we started.

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. The function f gives rise to a holomorphic map of Riemann surfaces $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$.

Before proving Theorem 1 we will prove a lemma.

Lemma 1. *Let f be a meromorphic function on a Riemann surface $\mathbb{C}$ with associated holomorphic map $F : X \to \mathbb{C}_\infty$.*

a. *If $p \in \mathbb{C}$ is a zero of f, then $mult_p(F) = ord_p(f)$.*
b. *If p is a pole of f, then $mult_p(F) = -ord_p(f)$.*
c. *If p is neither a zero nor a pole of f, then $mult_p(F) = ord_p(-f - f(p))$.*

Definition Let $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a nonconstant analytic map. A point $p \in X$ is a *ramification point* for F if $mult_p(F) \geq 2$. A point $y \in Y$ is a *branch point* for F if it is the image of a ramification point of F (cf., [8], page 45).

Definition A point c is a special point if there exists a corresponding ramification point.i

Thus, in our example, $c = 0$ and $c = \infty$ are special points, and $z = 0$ and $z = \infty$ are ramification points.

Part 3. **Our Problem**

7. Iteration

Iteration is the repeated composition of a function. Let us take a complex function $f(z)$. We would then have

$$\begin{aligned} f(z) &= \text{The first iterate of } f(z) \\ f(f(z)) = f^2(z) &= \text{The second iterate of } f(z) \\ f(f(f(z))) = f^3(z) &= \text{The third iterate of } f(z) \\ &\vdots \end{aligned}$$

and define $f^n(z)$ as the n^{th} iterate of $f(z)$.

Definition z_0 is called a *fixed point* of $f(z)$ if it satisfies $f(z_0) = z_0$ (cf. [5], page 20).

For example, let us take the function $f(z) = z^2$. If we let $z_0 = 0$ we have

$$\begin{aligned} z_0 &= 0 \\ f(0) &= 0 \\ f^{(2)}(0) &= 0 \\ &\vdots \\ f^{(n)}(0) &= 0 \end{aligned}$$

Regardless of the number of iterations we take of f, we will always stay fixed at $z_0 = 0$. We will not, however, always have fixed points. During iteration, we will also come across *periodic points*

Definition z_0 is called a *periodic point* of $f(z)$ if it satisfies $f^n(z_0) = z_0$ for some $n > 0$, with $f^k(z_0) \neq z_0$ for all $k < n$. We say that that z_0 is a *period n point* (cf., [5], page 20).

For example, let us take the function $f(z) = z^2 - 1$ and let $z_0 = 0$. We then have that

$$\begin{aligned} f(0) &= -1 \\ f^2(0) &= 0 \\ &\vdots \\ f^{2n}(0) &= 0 \\ f^{2n+1}(0) &= -1 \end{aligned}$$

Since in the above example we have that $f^2(0) = 0 = z_0$, $z_0 = 0$ is a period two point of our function.

8. The Permutations of the Periodic Points of the Family of Quadratic Polynomials

The primary goal of this paper is to look at the permutations of the periodic points of the quadratic family

$$f(z) = z^2 + c.$$

For each value of c, where $c \in \mathbb{C}$ is a parameter, we will have fixed points and periodic points of f. We need to think about our problem as if we were working with two different planes, the c-plane and the z-plane. Notice that the periodic points depend continuously on c. In the c plane we will have our c values, and in the z-plane we will have our corresponding fixed and periodic points. If we move c, our periodic points will move. If we let c travel along continuous, closed loops, our fixed and periodic points will travel along closed loops in the z-plane. In the c-plane, we will be traversing representatives, γ, of an element $\langle\gamma\rangle$ from the fundamental group π_i of a certain open region $X \subset \mathbb{C}$.

For all $t \in [0,1]$ and for each $n \in \mathbb{N}$ we define the set

$$P_n^t = \left\{ z \middle| \; z \text{ is a period } n \text{ point of } z^2 + c_t \right\}$$

Where

$$\gamma : [0,1] \to \mathbb{C}$$

and

$$c_t = \gamma(t)$$

If the loop γ is closed, that is

$$c_0 = c_1$$

then we have that

$$P_n^0 = P_n^1$$

So γ induces a permutation on the set P_n^0. We will be mapping the fundamental group $\pi_1(X,p)$ to the group of permutations of the periodic points, which we will call $\mathbf{G}$, in the z-plane. We will denote this mapping, φ, as

$$\varphi : \pi_1(X,p) \to \mathbf{G}.$$

We have that φ is operation preserving. That is

$$\varphi(\langle\gamma_1\rangle) \cdot \varphi(\langle\gamma_2\rangle) = \varphi(\langle\gamma_1 \cdot \gamma_2\rangle).$$

Then φ is a group homomorphism. Therefore, $\varphi(\pi_1(X,p))$ is a subgroup of $\mathbf{G}$, with generators $\varphi(\langle\gamma_i\rangle)$. The identity element of $\mathbf{G}$ is $\varphi(\langle\gamma_0\rangle)$, where γ_0 is the trivial loop.

For each n we will have m period n points. We want to number these periodic points from 1 to m. Since we are studying the permutation of these points, it is helpful to use cycle notation. For example, say we are working with three periodic points, which we number as 1, 2, and 3. Suppose the permutation φ on these gives

$$\varphi(1) = 2, \quad \varphi(2) = 3, \quad \varphi(3) = 1.$$

The permutation φ takes $1 \to 2, 2 \to 3$ and $3 \to 1$. Using cycle notation, we can write

$$(123).$$

We read cycle notation from left to right, so we still have that φ takes $1 \to 2, 2 \to 3$ and $3 \to 1$.

Let us define a permutation β on the same set of periodic points as

$$(132).$$

Say we wanted to know the permutation of first taking β and then φ. To multiply using cyclic notation, we read right to left. That is, we have

$$\varphi \cdot \beta = (123)(132)$$

Inside the parenthesis, however, we continue to read left to right. That is, to find the permutation of 1, we start on the right and have that 1 goes to 3. Moving left we then see that 3 goes to 1. So $1 \to 1$. In a similar manner we have $2 \to 2$, and $3 \to 3$.

We will begin our work of the permutation of the periodic points by starting with the fixed points and then moving on to the period-2, period-3 and period-4 cases.

Part 4. Research

9. The Permutations of the Fixed Points

Let us look at the fixed points of $f(z) = z^2 + c$, that is, the values of z that solve the equation

$$z^2 + c = z,$$

where $c, z \in \mathbb{C}$. If we rearrange the equation, we then have that

$$\begin{aligned} f(z) &= z \\ z^2 - z + c &= 0. \end{aligned}$$

Let us let

$$Q_1 = z^2 - z + c.$$

We are then solving for the zeros of Q_1. From the Fundamental Theorem of Algebra, we know that our Q_1 will have at most two zeros in the complex plane, when counted according to multiplicity. So for each value of c that we choose in the complex plane, we will have two fixed points. From the Chapter on Riemann Surfaces, we know that the points that affect our permutations are the special points of a function f.

We get special points for values of c such that z is a ramification point, and our branch point is $Q_1(z) = 0$. We can find the special points of our function in two different ways.

9.1. Finding the Special Points of the First Iterate of the Quadratic Using the Quadratic Formula. Since we are looking at a quadratic, we know that we will have at most two zeros for each value of c that we choose. Using the quadratic formula, these two zeros are in the form

$$z = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$

We notice that the value of z will be of multiplicity two when the discriminant of the quadratic formula is zero. We then want,

$$\begin{aligned} 1 - 4c &= 0 \\ c &= \frac{1}{4}. \end{aligned}$$

Hence, when we have $c = \frac{1}{4}$ we have a ramification point at $z = \frac{1}{2}$.

9.2. **Finding the Special Points of the First Iterate of the Quadratic Using Derivatives.** Let us suppose that a is a zero of multiplicity m of a polynomial $P(z)$. We then have that

$$P(z) = (z-a)^m P_m(z)$$

with $P_m(z) \neq 0$, and $P(a) = 0$. Let us successively take the derivative of $P(z)$, and evaluate at a. We have that

$$\begin{aligned} P'(a) &= 0 \\ &\vdots \\ P^{(m-1)}(a) &= 0 \\ P^{(m)}(a) &\neq 0. \end{aligned}$$

Thus, the multiplicity of a zero is the order of the first nonvanishing derivative (cf. [1], page 43).

We can now apply the concepts of derivatives to find a zero of multiplicity two. We want $Q_1'(a) = 0$, and $Q_1''(a) \neq 0$.

$$\begin{aligned} Q_1(z) &= z^2 - z + c \\ Q_1'(z) &= 2z - 1 \end{aligned}$$

Solving $Q_1'(z) = 0$, we find that we have $z = \frac{1}{2}$. Substituting back into our original equation yields $c = \frac{1}{4}$, which we had found before.

For the fixed points, we then have a single special point $c = \frac{1}{4}$ with corresponding ramification point $z = \frac{1}{2}$. Let us denote $c_1^* = \frac{1}{4}$. Let $\langle\gamma_1\rangle$ be the homotopy class of loops that go around c_1^* once that can be deformed into simple closed curves taken in the counterclockwise direction, and $\langle\gamma_0\rangle$ is the homotopy class corresponding to the trivial loop. These two classes are the two elements are the identity and the generator in the fundamental group of $X = \mathbb{C} - \{c_1^*\}$ based at p and denoted $\pi_1(X, p)$. Since our choice of p is irrelevant, we denote the fundamental group as $\pi_1(X)$. Since for all $c \in X = \mathbb{C} - c_1^*$, we will have two distinct zeros, we will number our fixed points at $c = p$ as 1 and 2. The permutation of the fixed points, $\mathbf{G}$, is then the group of permutations of two elements, S_2. We have then then that

$$\varphi : \pi_1(X) \to S_2$$

Let us look at the permutations induced by $\varphi(\langle\gamma_0\rangle)$ and $\varphi(\langle\gamma_1\rangle)$ by choosing a base point of $p = 1$.

9.3. **The Permutations Induced by $\langle\gamma_0\rangle$.** Since $\langle\gamma_0\rangle$ does not go around c_1^*, our permutation should be the identity. Using $p = 1$ as our base point, we can construct the loop

$$c = .75 + .25e^{2\pi it}$$

in the c plane. By increasing t from 0 to 1, we traverse a circle. When $t = 0$ or 1, we have $c = 1$ and two distinct roots. We want to be able to visually see in the z-plane these roots. We will number these roots 1 and 2. Looking at Figure 3 we have our γ_0 on the left-hand side, and the permutation induced by $c = .75 + .25e^{2\pi it}$. Notice that indeed our permutation is the identity.

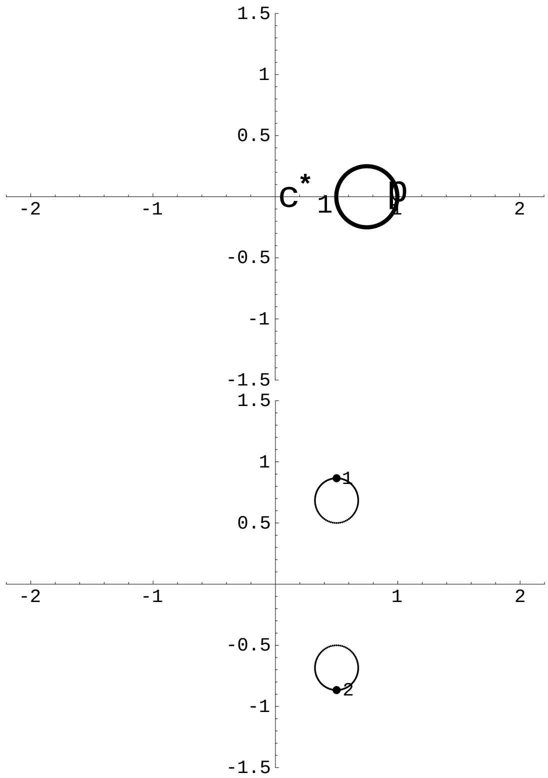

FIGURE 3. c-plane, z-plane

Hence,

$$\varphi(\langle\gamma_0\rangle) = (1)(2) = \mathrm{Id}$$

9.4. **The Permutations Induced by $\langle\gamma_1\rangle$.** Keeping $p = 1$ as our base point, we want to construct a loop from the homotopy class $\langle\gamma_1\rangle$. Let c travel along the unit circle. Our geometric representations of the parameter space and our mapping are shown in Figure 4.

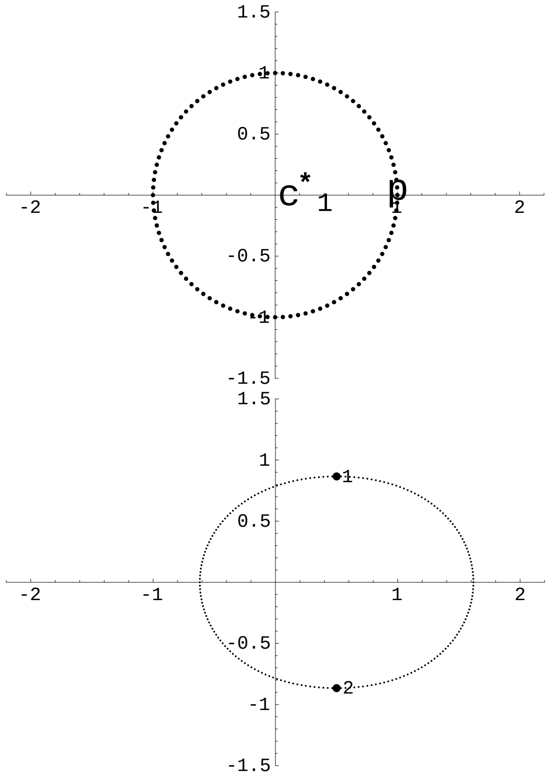

FIGURE 4. c-plane, z-plane

We now notice that our fixed points lie on the same loop. Hence as our c-value travels along the unit circle in the c-plane, the permutation of our fixed points at $p = 1$ is $1 \to 2, 2 \to 1$ in the z-plane. Using cyclic notation, we have that

$$\varphi(\langle\gamma_1\rangle) = (12)$$

9.5. **Summary of the Permutations of the Fixed Points.** In summary, φ is a mapping from our topological group $\pi_1(X, p)$, where $X = \mathbb{C} - \left\{\frac{1}{4}\right\}$, to the permutation of two elements, S_2. Our two elements of $\varphi(\pi_1(X, p))$ are

$$\begin{aligned}
\varphi(\langle\gamma_0\rangle) &= (1)(2) = \mathrm{Id} \\
\varphi(\langle\gamma_1\rangle) &= (12)
\end{aligned}$$

10. THE PERMUTATIONS OF PERIOD TWO POINTS

Let us now look at period two points of $f(z) = z^2 + c$.

$$\begin{aligned}
f^2(z) &= f(z^2 + c) = (z^2 + c)^2 + c \\
&= z^4 + 2cz^2 + c^2 + c.
\end{aligned}$$

For the period two points of $f(z)$, we want to have $f^2(z) = z$, that is

$$z^4 + 2cz^2 - z + c^2 + c = 0.$$

Since $f^2(z)$ is of degree four in z, we know that $f^2(z) - z$ has exactly four solutions when counted according to multiplicity. Our work in finding the special points, however, is not as difficult as it may seem. A fixed point will also be a solution to $f^2(z) = 0$. We can then factor $f(z) - z$ into $f^2(z) - z$. Let us define $Q_2(z)$, with 2 representing the second iteration of the quadratic, as

$$Q_2(z) = \frac{f^2(z) - z}{f(z) - z}.$$

By completing long division, the polynomial quotient, we get

$$Q_2(z) = z^2 + z + c + 1$$

which is another quadratic.

Using the same methods from the last section, we find that there is only one special point for a period two point. It is when $c = -\frac{3}{4}$.

Similarly to the permutation of fixed points, let us denote the special point by $c_1^* = -\frac{3}{4}$. We then have the class of identity loops as $\langle\gamma_0\rangle$ that do not contain c_1^*, and the homotopy class $\langle\gamma_1\rangle$ that does contain c_1^* and winds around c_1^* exactly once. So $\langle\gamma_0\rangle$ and $\langle\gamma_1\rangle$ are elements of the fundamental group $\pi_1(X, p)$, where $X = \mathbb{C} - \left\{-\frac{3}{4}\right\}$.

10.1. **The Permutations Induced by $\langle\gamma_0\rangle$.** Let us let $p = 1$ and observe the permutations of the period two points by taking $p = 1$. When $c = p$, we will call our two periodic points, 1 and 2. Again, since the homotopy class $\langle\gamma_0\rangle$ does not contain the ramification point of our second iteration of the quadratic, our permutation should be the identity. Let us construct a loop in $\langle\gamma_1\rangle$ as

$$c = .75 + .25e^{2\pi it}$$

in the c plane. If we look at the mapping of the permutations of period two points in Figure 5, we notice that indeed our permutation is the identity.

Hence,

$$\varphi(\langle\gamma_0\rangle) = (1)(2) = \mathrm{Id}$$

10.2. **The Permutations Induced by $\langle\gamma_1\rangle$.** We will keep $p = 1$. We now want to construct a loop from the homotopy class $\langle\gamma_1\rangle$ that contains c_1^* Let us let c travel along the unit circle centered at the origin. Our geometric representation of our mapping is shown in Figure 6.

We then have that

$$\varphi(\langle\gamma_1\rangle) = (12)$$

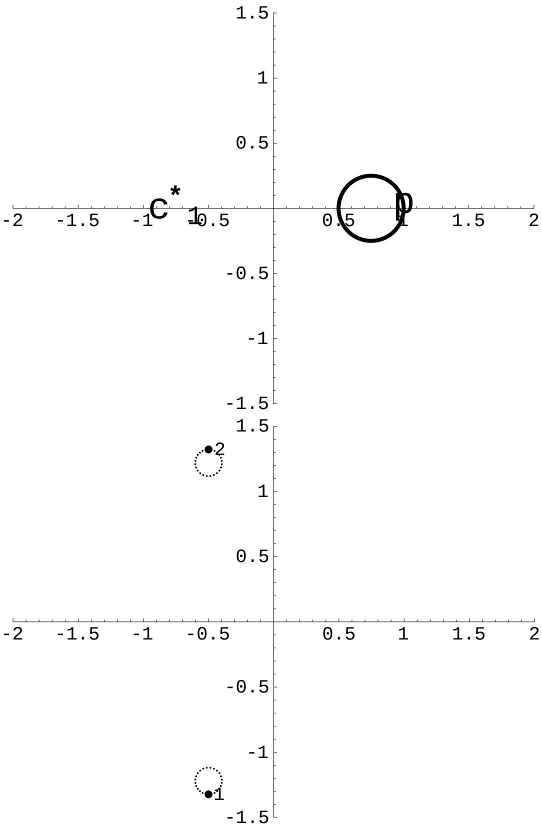

FIGURE 5. c-plane, z-plane

10.3. **Summary of the Permutations of the Period Two Points.** In summary, the second iteration has a ramification point where $c_1^* = -\frac{3}{4}$. Our punctured plane is then $X = \mathbb{C} - \{c_1^*\}$. We then have φ as a mapping from our group $\pi_1(X, p)$, to the permutation of two elements, S_2. Our two elements of $\varphi(\pi_1(X, p))$ are

$$\begin{aligned} \varphi(\langle\gamma_0\rangle) &= (1)(2) = \mathrm{Id} \\ \varphi(\langle\gamma_1\rangle) &= (12) \end{aligned}$$

The permutation of period two points is quite similar to the permutations of the fixed points of the quadratic. The permutations of periodic points begins to become interesting when we start looking at the permutations of period three points.

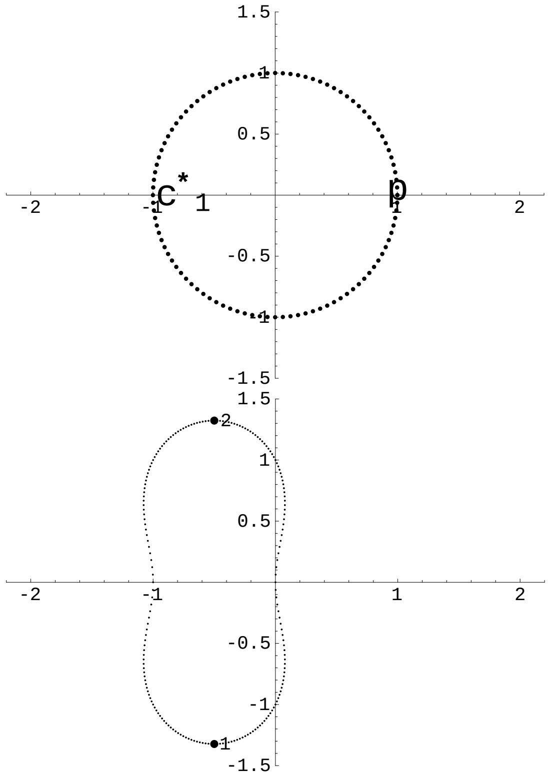

FIGURE 6. c-plane, z-plane

11. The Permutations of Period Three Points

Let us now begin working with the third iteration of $f(z)$:

$$f^3(z) = ((z^2+c)^2+c)^2+c = z^8+4cz^6+\left(2c+6c^2\right)z^4+\left(4c^2+4c^3\right)z^2 + c^4+2c^3+c^2+c$$

Hence, we are working with a polynomial of degree eight. Similar to the second iteration, fixed points of a polynomial satisfy $f_3(z) = z$. We can then divide $f^3(z) - z$ by $f(z) - z$.

$$Q_3 = \frac{f^3(z) - z}{f(z) - z} = z^6+z^5+(1+3c)\,z^4+(1+2c)\,z^3+\left(1+3c+3c^2\right)z^2 + \left(1+2c+c^2\right)z+c^3+2c^2+c+1$$

Our quotient, however, is a polynomial of degree six. To get a better sense of the permutations of period three points, let c travel along a variety of curves and examine their mappings in the z-plane. See Figure 8.

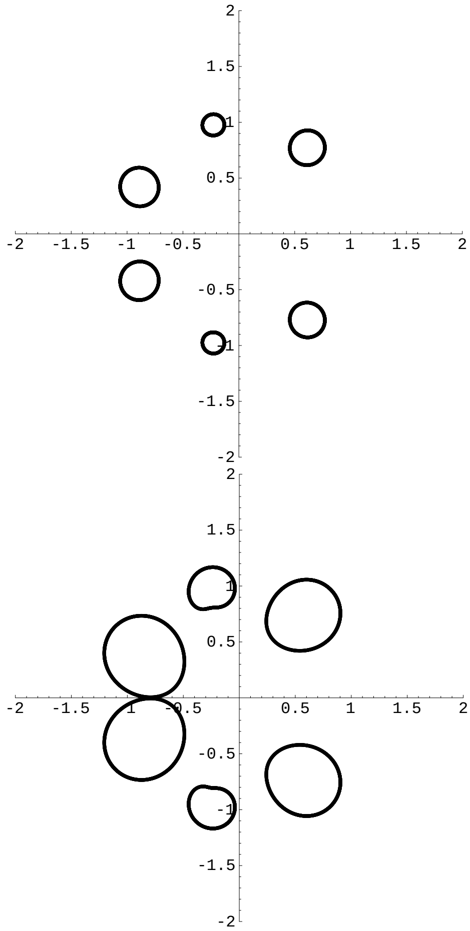

FIGURE 7. z-plane

As we can see, our function is much more complicated than it was for either the period two or the fixed points. Since we are working with a polynomial of degree six, we cannot find our ramification points using the quadratic formula. Instead, we will find them using derivatives.

11.1. **Finding the Special Points of the Third Iterate of the Quadratic.** To find a zero, a, of multiplicity, $m > 1$, a must satisfy

$$\begin{aligned} P(a) &= 0 \\ P'(a) &= 0 \\ &\vdots \\ P^{m-1}(a) &= 0 \\ P^{m}(a) &\neq 0. \end{aligned}$$

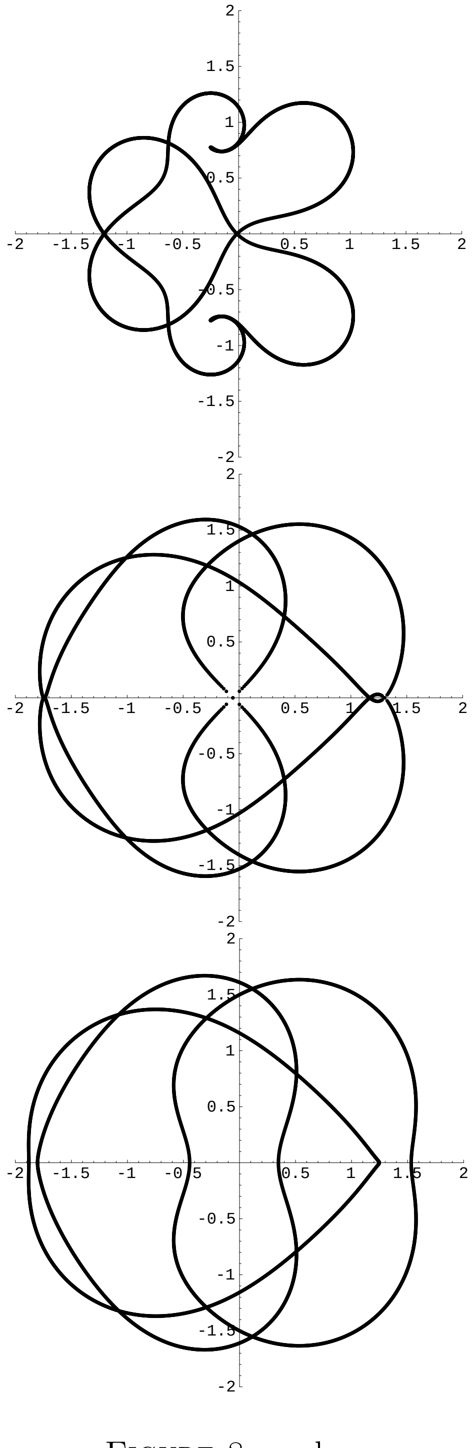

FIGURE 8. z-plane

To find the ramification points of the third iteration of $f(z)$, we need to first find the successive derivatives of $Q_3(z)$. To have a zero of multiplicity six we would have to have z and c values such that

$$(Q_3)^{(5)} = (Q_3)^{(4)} = (Q_3)^{(3)} = (Q_3)'' = (Q_3)' = (Q_3) = 0.$$

Using Mathematica to solve simultaneous equations, we have no solution. Hence there is no period three point of multiplicity six. We then try to determine if the third iterate has a period three point of multiplicity five. Following in a similar manner yields,

$$(Q_3)^{(4)} = (Q_3)^{(3)} = (Q_3)'' = (Q_3)' = (Q_3) = 0.$$

Again, we have no solution. In fact, it is not until

$$(Q_3)'' = (Q_3)' = (Q_3) = 0$$

that we find our first multiple zero at $c = \frac{1}{8}\left(-1 \pm 3i\sqrt{3}\right)$. We then have zeros of multiplicity three at $z = \frac{1}{4}\left(-1 \pm i\sqrt{3}\right)$. Continuing further, we also find that we have zeros of multiplicity two when $c = -\frac{7}{4}$.

Hence, the ramification points of the third iterate of $f(z)$ are $z = \frac{1}{4}$ and $z = -\frac{7}{8} \pm 3i\sqrt{3}$.

11.2. **The Fundamental Group for the Third Iterate of the Quadratic.** In the case of the fixed points and the period two points, we only had the identity $\langle\gamma_0\rangle$ and one generator, $\langle\gamma_1\rangle$, in the fundamental group.

Let

$$\begin{aligned} c_1^* &= -\frac{7}{4} \\ c_2^* &= -\frac{1}{8}(-1 + 3i\sqrt{3}) \\ c_3^* &= -\frac{1}{8}(-1 - 3i\sqrt{3}). \end{aligned}$$

Then $X = \mathbb{C} - \{-\frac{7}{4}, -\frac{1}{8}\left(-1 + 3i\sqrt{3}\right), -\frac{1}{8}\left(-1 - 3i\sqrt{3}\right)\}$. Choose a base point $p = -\frac{1}{8}$. We then have that $\langle\gamma_1\rangle$ goes around c_1^*, $\langle\gamma_2\rangle$ goes around c_2^* and $\langle\gamma_3\rangle$ goes around c_3^*. So $\langle\gamma_1\rangle, \langle\gamma_2\rangle$ and $\langle\gamma_3\rangle$ are elements of the the group $\pi_1(X, p)$. We now want to determine the permutations of the period three points. We will first look at the permutations generated by the primary generators and then examine some of the products.

11.3. **The Permutations Induced by $\langle\gamma_0\rangle$.** Like the permutations of the fixed points and the period two points, loops contained in the homotopy class $\langle\gamma_0\rangle$ represent the identity. That is, a loop that does not go around any of the special points. Let

$$c = \gamma_0(t) = -\frac{2}{8} + \frac{1}{8}e^{2\pi i t}.$$

In Figure 9 we have the geometric representation of the permutation of our period three points.

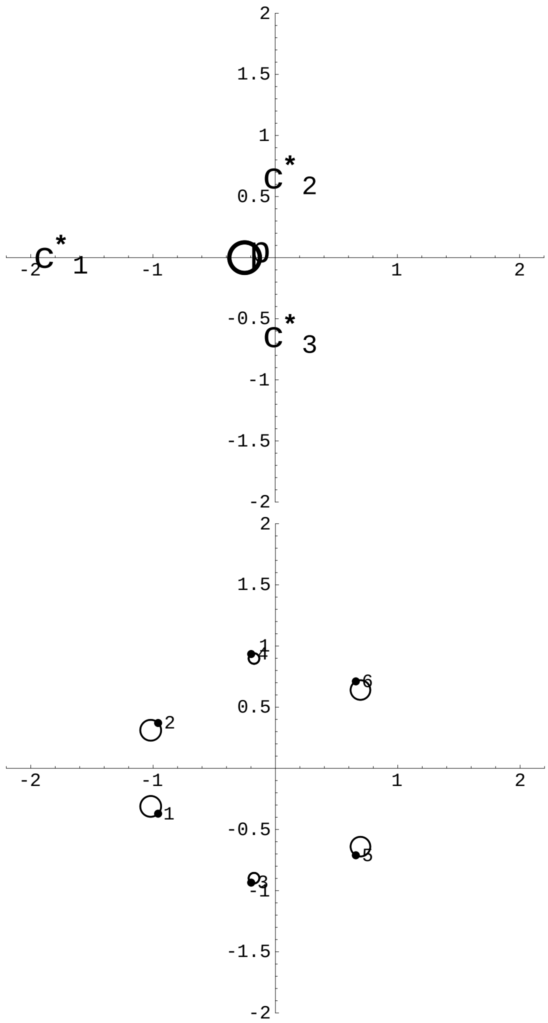

FIGURE 9. c-plane, z-plane

Indeed,

$$\varphi(\langle\gamma_0\rangle) = (1)(2)(3)(4)(5)(6) = \mathrm{Id}.$$

11.4. **The Permutations Induced by $\langle\gamma_1\rangle$.** We want to create a $\gamma_1 \in \langle\gamma_1\rangle$. Since our base point is $p = -\frac{1}{8}$, we want to travel closer to c_1^*, take a circle around c_1^*, and then travel back to to our base point. $\gamma_1(t)$ then becomes,

$$\gamma_1(t) = \begin{cases} c(t) = -\frac{1}{8} + \left(-\frac{21}{8}\right)t, & 0 \le t \le \frac{1}{3}; \\ c(t) = -1.75 + .75e^{6\pi i\left(t-\frac{1}{3}\right)}, & \frac{1}{3} \le t \le \frac{2}{3}; \\ c(t) = -\frac{7}{8} + \left(\frac{9}{4}\right)\left(t - \frac{2}{3}\right), & \frac{2}{3} \le t \le 1. \end{cases}$$

For each c value we will have six z values. We are most concerned about the zeros of $Q(z)$ for $c = -\frac{1}{8}$, which is our base point. For each c value we are following the paths of these z values. A complete graph is shown in Figure 10. From our graph, we notice that we have the following permutation.

$$\varphi(\langle\gamma_1\rangle) = (12)(34)(56).$$

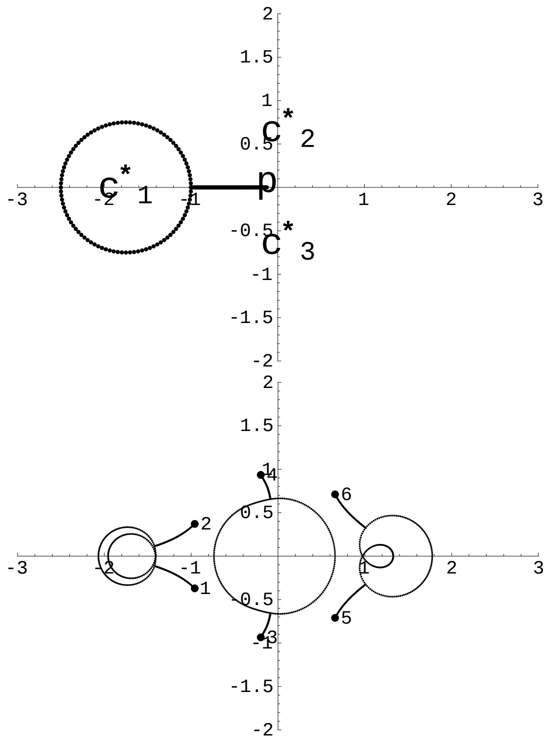

FIGURE 10. c-plane, z-plane

11.5. **Permutations Induced by $\langle\gamma_2\rangle$.** Loops in the homotopy class $\langle\gamma_2\rangle$ contain the point $c_2^* = -\frac{1}{8}\left(-1+3i\sqrt{3}\right)$. Since $p = -\frac{1}{8}$, we can then take a circle centered at c_2^*. Figure 11 shows the z-plane, with a gap to show the z values for our base point, and then the complete mapping.

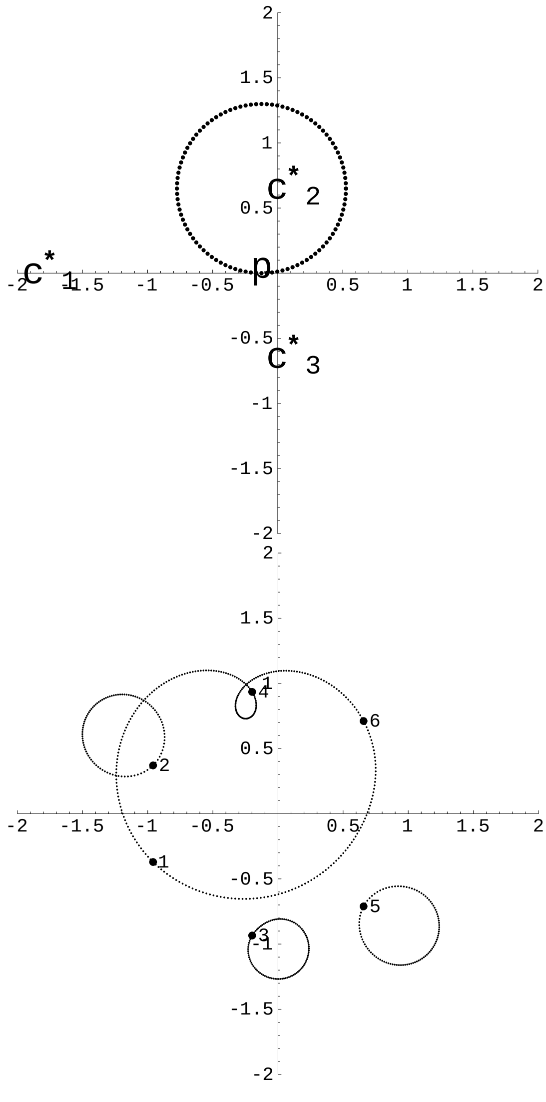

FIGURE 11. c-plane, z-plane

Using the same technique as in the pervious section, let us label the six period three points at our base point. As we vary c, we have that

$$\varphi(\langle\gamma_2\rangle) = (146).$$

11.6. **Permutations Induced by $\langle\gamma_3\rangle$.** Loops in the homotopy class $\langle\gamma_3\rangle$ contain the point $c_2^* = -\frac{1}{8}\left(-1 - 3i\sqrt{3}\right)$. Since $p = -\frac{1}{8}$, we can then take a circle centered at c_2^*. Figure 12 shows the z-plane, with a gap to show the z values for our base point, and then the complete mapping.

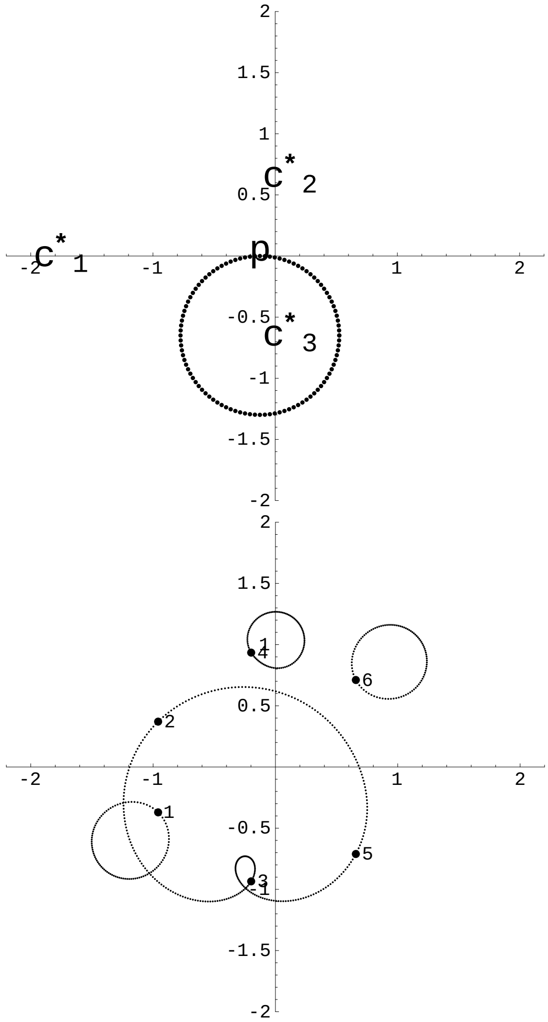

FIGURE 12. c-plane, z-plane

Using the same technique as in the previous section, let us label the six period three points at our base point. As we vary c, we have that

$$\varphi(\langle\gamma_3\rangle) = (253).$$

11.7. **Permutations Induced by Products of the Generators.** We now have that

$$\begin{aligned}\varphi(\langle\gamma_1\rangle) &= (12)(34)(56)\\ \varphi(\langle\gamma_2\rangle) &= (146)\\ \varphi(\langle\gamma_3\rangle) &= (253)\end{aligned}$$

Based on the permutations of our three homotopy classes we can begin finding products of the generators. However, when we find the products we need to keep in mind that our group is not abelian. Thus, the order in which we take our loops matters. Also, while we can computationally find the products, we need to pay attention to our equivalence class of loops and what we can actually deform into.

For example, let us take a loop that contains c_1^*, c_2^* and c_2^*. We can find the permutation of this loop by taking the product of $\varphi(\langle\gamma_1\rangle)$, $\varphi(\langle\gamma_2\rangle)$ and $\varphi(\langle\gamma_2\rangle)$. There are six different types of orders we could take, but only some orders are the ones we want. Remember, in order for

$$\langle\gamma_1 \cdot \gamma_2 \cdot \gamma_3\rangle$$

to be equivalent to

$$\langle\gamma_1\rangle \cdot \langle\gamma_2\rangle \cdot \langle\gamma_3\rangle$$

we need to be able to deform the product into the larger loop. We can only do that when we take the product as

$$\langle\gamma_2\rangle \cdot \langle\gamma_1\rangle \cdot \langle\gamma_3\rangle = \langle\gamma_3\rangle \cdot \langle\gamma_1\rangle \cdot \langle\gamma_2\rangle$$

Hence we have that a loop that contains all three ramification points is of the homotopy class $\langle\gamma_2 \cdot \gamma_1 \cdot \gamma_3\rangle = \langle\gamma_3 \cdot \gamma_1 \cdot \gamma_2\rangle$.

We then have that the mapping of three homotopy class onto permutations is

$$\begin{aligned}\varphi(\langle\gamma_2 \cdot \gamma_1 \cdot \gamma_3\rangle) &= \varphi(\langle\gamma_2\rangle) \cdot \varphi(\langle\gamma_1\rangle) \cdot \varphi(\langle\gamma_3\rangle)\\ &= (146)(12)(34)(56)(253)\\ &= (12)(34)(56)\end{aligned}$$

We can check our manual computation with the computer by taking a loop, base $p = -\frac{1}{8}$.

$$\gamma_1(t) = \begin{cases} c(t) = -\frac{1}{8} + \left(\frac{51}{8}\right)t, & 0 \le t \le \frac{1}{3};\\ c(t) = 2e^{6\pi i\left(t-\frac{1}{3}\right)}, & \frac{1}{3} \le t \le \frac{2}{3};\\ c(t) = 2 + \left(-\frac{45}{8}\right)\left(t - \frac{2}{3}\right), & \frac{2}{3} \le t \le 1.\end{cases}$$

We then have the geometric representation of our paths in the z-plane in Figure 13.

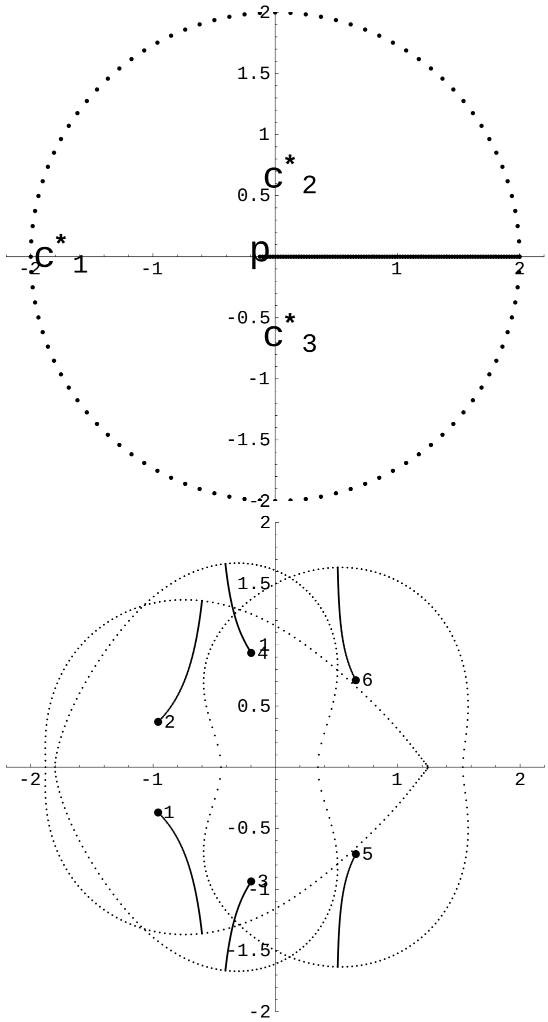

FIGURE 13. c-plane, z-plane

Let us take a loop that contains c_2^* and c_3^*. We notice that

$$\varphi\left(\langle\gamma_2\rangle\right)\cdot\varphi\left(\langle\gamma_3\rangle\right)=(146)(253)$$

and

$$\varphi\left(\langle\gamma_3\rangle\right)\cdot\varphi\left(\langle\gamma_2\rangle\right)=(253)(146).$$

So we have $\varphi(\gamma_2\cdot\gamma_3)=\varphi(\gamma_3\cdot\gamma_2)=(146)(253)$. Again we can visually represent the permutation the permutations in the z-plane.

$$\gamma_1(t)=\begin{cases} c(t)=-\frac{1}{8}+\left(\frac{27}{8}\right)t, & 0\le t\le\frac{1}{3};\\ c(t)=e^{6\pi i\left(t-\frac{1}{3}\right)}, & \frac{1}{3}\le t\le\frac{2}{3};\\ c(t)=1+\left(-\frac{27}{8}\right)\left(t-\frac{2}{3}\right), & \frac{2}{3}\le t\le 1.\end{cases}$$

Which gives the graphs found in Figure 14.

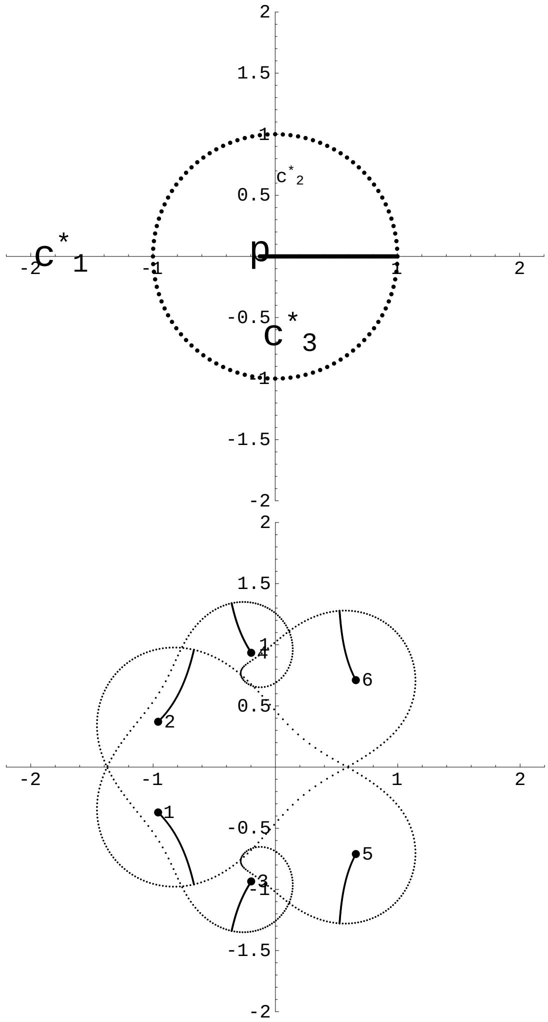

FIGURE 14. c-plane, z-plane

Let us now take a loop that goes around c_1^* and c_2^*. The class of loops that contain both points is a product of $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$. If we draw these two loops in the c-plane, we notice that we can only deform them into a larger loop if we first take $\langle\gamma_2\rangle$ and then take $\langle\gamma_1\rangle$. We then have that

$$
\begin{aligned}
\varphi(\langle\gamma_1 \cdot \gamma_2\rangle) &= \varphi(\langle\gamma_1\rangle \cdot \langle\gamma_2\rangle) \\
&= \varphi(\langle\gamma_1\rangle) \cdot \varphi(\langle\gamma_2\rangle) \\
&= (12)(34)(56)(146) \\
&= (134562).
\end{aligned}
$$

Let us take a loop with a base point of $p = -\frac{1}{8}$ and geometrically view our mapping onto the z-plane. We want to first move our base point along a line, $\langle\gamma_0\rangle$, then take a circle about both c_1^* and c_2^*, and then trace our path back to our base point. We have the following values for c:

$$
\gamma_1(t) = \begin{cases} c(t) = -\frac{1}{8} + \left(\frac{21}{40} + 3i\right) t, & 0 \le t \le \frac{1}{3}; \\ c(t) = (-1.75 + i) + 1.8e^{6\pi i\left(t - \frac{1}{3}\right)}, & \frac{1}{3} \le t \le \frac{2}{3}; \\ c(t) = \left(\frac{7}{40} +\right) + \left(-\frac{9}{10} - 3i\right)\left(t - \frac{2}{3}\right), & \frac{2}{3} \le t \le 1. \end{cases}
$$

Our geometric representation of the permutation of the periodic points are shown in Figure 15

Lastly, let us take a loop that contains c_1^* and c_3^*. We then have that

$$
\begin{aligned}
\varphi(\langle\gamma_1 \cdot \gamma_3\rangle) &= \varphi(\langle\gamma_1\rangle \cdot \langle\gamma_3\rangle) \\
&= \varphi(\langle\gamma_1\rangle) \cdot \varphi(\langle\gamma_2\rangle) \\
&= (12)(34)(56)(253) \\
&= (126543).
\end{aligned}
$$

Which we can view visually in Figure 16.

11.8. **Summary of the Permutations of the Period Three Points.** For the third iteration of our quadratic, we have special points at

$$
\begin{aligned}
c_1^* &= -\frac{7}{4} \\
c_2^* &= -\frac{1}{8}(-1 + 3i\sqrt{3}) \\
c_3^* &= -\frac{1}{8}(-1 - 3i\sqrt{3}).
\end{aligned}
$$

The fundamental group of the third iteration is on the topological space $X = \mathbb{C} - \{c_1^*, c_2^*, c_3^*\}$. The identity and generators of π_1 are $\langle\gamma_0\rangle, \langle\gamma_1\rangle, \langle\gamma_2\rangle$, and $\langle\gamma_3\rangle$.

We then have φ as a mapping from our group $\pi_1(X, p)$, to the permutation of six roots. Our elements of $\varphi\left(\pi_1(X, p)\right)$ are

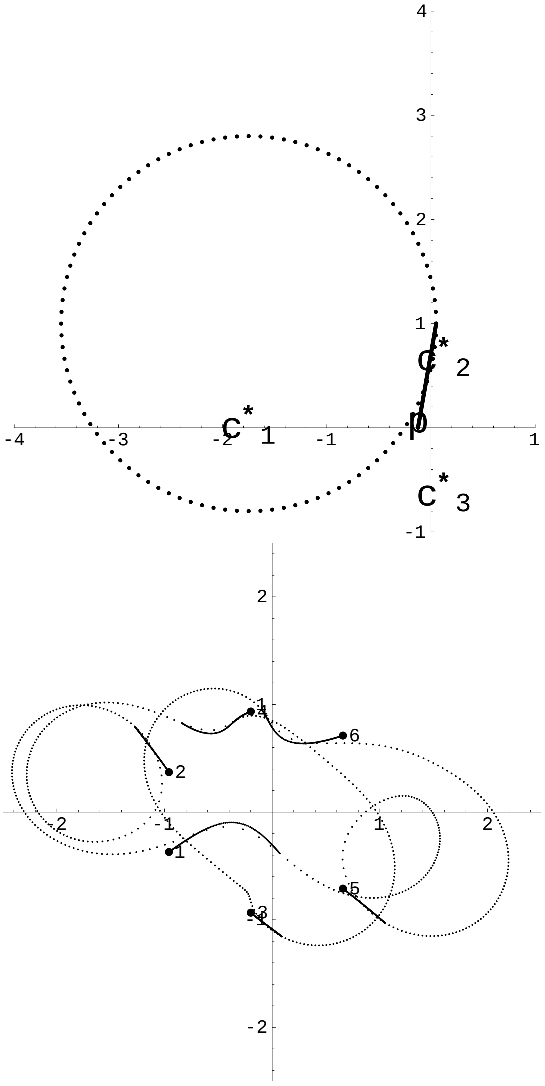

FIGURE 15. c-plane, z-plane

$$
\begin{aligned}
\varphi\left(\langle\gamma_0\rangle\right) &= (1)(2)(3)(4)(5)(6)\\
\varphi\left(\langle\gamma_1\rangle\right) = \varphi\left(\langle\gamma_2\rangle\cdot\langle\gamma_1\rangle\cdot\langle\gamma_3\rangle\right) = \varphi\left(\langle\gamma_3\rangle\cdot\langle\gamma_1\rangle\cdot\langle\gamma_2\rangle\right) &= (12)(34)(56)\\
\varphi\left(\langle\gamma_2\rangle\right) &= (146)\\
\varphi\left(\langle\gamma_3\rangle\right) &= (253)\\
\varphi\left(\langle\gamma_1\rangle\cdot\langle\gamma_2\rangle\right) &= (134562)\\
\varphi\left(\langle\gamma_1\rangle\cdot\langle\gamma_3\rangle\right) &= (126543)\\
\varphi\left(\langle\gamma_2\rangle\cdot\langle\gamma_3\rangle\right) = \varphi\left(\langle\gamma_3\rangle\cdot\langle\gamma_2\rangle\right) &= (146)(253)
\end{aligned}
$$

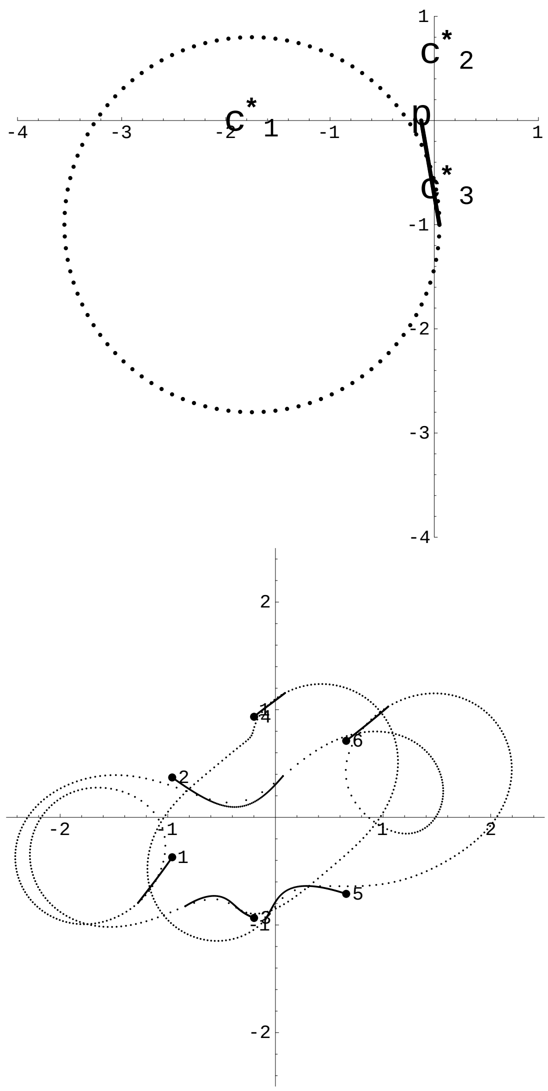

FIGURE 16. c-plane, z-plane

12. The Permutations of Period Four Points

The permutations of the period four points of our quadratic are the last permutations we will be studying in this paper. Even though not all of the permutations of the period four points are understood, we will have a chance to at least begin to have a taste of them. Quickly, we have noticed that our permutations are growing complex quite rapidly. Let us view some mappings, in Figure 17, in the z-plane to visualize the complexities of the fourth iteration.

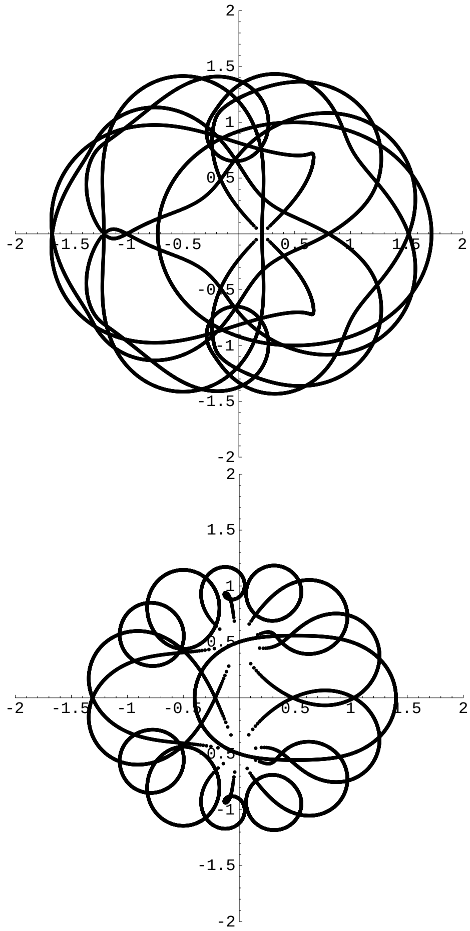

FIGURE 17. $c = \left(-\frac{5}{4}\right) e^{2\pi i t}, c = (.25 - .5i)\, e^{2\pi i t}$

The fourth iteration of our quadratic is

$$\begin{aligned}
f^4(z) &= (((z^2+c)^2+c)^2+c)^2+c \\
&= c + c^2 + 2\,c^3 + 5\,c^4 + 6\,c^5 + 6\,c^6 + 4\,c^7 + c^8 \\
&+ \left(8\,c^3 + 16\,c^4 + 24\,c^5 + 24\,c^6 + 8\,c^7\right)\, z^2 \\
&+ \left(4\,c^2 + 16\,c^3 + 36\,c^4 + 60\,c^5 + 28\,c^6\right)\, z^4 \\
&+ \left(8\,c^2 + 24\,c^3 + 80\,c^4 + 56\,c^5\right)\, z^6 \\
&+ \left(2\,c + 6\,c^2 + 60\,c^3 + 70\,c^4\right)\, z^8 + \left(24\,c^2 + 56\,c^3\right)\, z^{10} \\
&+ \left(4\,c + 28\,c^2\right)\, z^{12} + 8\,c\,z^{14} + z^{16}.
\end{aligned}$$

Whereas in the previous cases, we could divide our function minus z by $f_1 - z$, in the fourth iterate, we can also factor the period two points. We then have that

$$\begin{aligned}
Q_4 &= \frac{f^4(z)-z}{f^2(z)-z} \\
&= 1+2\,c^2+3\,c^3+3\,c^4+3\,c^5+c^6+\left(2\,c+c^2+2\,c^3+c^4\right)\,z \\
&+ \left(c+5\,c^2+6\,c^3+12\,c^4+6\,c^5\right)\,z^2+\left(1+4\,c^2+4\,c^3\right)\,z^3 \\
&+ \left(4\,c+3\,c^2+18\,c^3+15\,c^4\right)\,z^4+\left(2\,c+6\,c^2\right)\,z^5 \\
&+ \left(1+12\,c^2+20\,c^3\right)\,z^6+4\,c\,z^7+\left(3\,c+15\,c^2\right)\,z^8+z^9 \\
&+ 6\,c\,z^{10}+z^{12}.
\end{aligned}$$

We are working with a polynomial of degree twelve, hence we will have at most twelve roots when counted according to their multiplicity.

12.1. **Finding the Special Points for the Fourth Iterate of the Quadratic.** To find the special points, we want to use our knowledge of derivatives in the same manner as we used to find the special points for the third iteration of the quadratic. Using Mathematica, we find that we have special points at

$$\begin{aligned}
c_1^* &= \frac{3}{4}\left(-1-2^{\frac{2}{3}}\right) \\
c_2^* &= -\frac{5}{4} \\
c_3^* &= -\frac{3}{4}+\frac{3\left(1+i\sqrt{3}\right)}{4\,2^{\frac{1}{3}}} \\
c_4^* &= -\frac{3}{4}+\frac{3\left(1-i\sqrt{3}\right)}{4\,2^{\frac{1}{3}}} \\
c_5^* &= \frac{1}{4}+\frac{i}{2} \\
c_6^* &= \frac{1}{4}-\frac{i}{2}
\end{aligned}$$

We have that if we let $c = c_1^*, c_2^*, c_3^*$, or $, c_4^*$, we will have roots of multiplicity two. If we let $c = c_5^*$ or c_6^*, we have roots of multiplicity four.

We now have six homotopy classes of generators, that is $\langle\gamma_1\rangle \ldots \langle\gamma_6\rangle$. These generators are elements of the fundamental group on the topological plane $X = \mathbb{C} - \{c_1^*, c_2^*, c_3^*, c_4^*, c_5^*, c_6^*\}$, based at a point $p \in \mathbb{C}$, and denoted as $\pi_1(X, p)$. For the fundamental group, let us let p be the origin. We can then examine the mapping of $\varphi\left(\pi_1(X, 0)\right)$ to the permutations of our period four points. To find each permutation, we have p travel along a line, go around a circle that contains our ramification point, and then return to p, which is the same method we used for period three permutations. We will look closely at $\varphi(\langle\gamma_1\rangle)$ and $(\langle\gamma_2\rangle)$ to become comfortable with our method, and then just state the other permutations for the period four points.

12.2. **Permutations Induced by $\langle\gamma_1\rangle$.** We have that $\langle\gamma_1\rangle$ is the homotopy class of loops that contains the ramification point $c_1^* = \frac{3}{4}\left(-1-2^{\frac{2}{3}}\right)$ once in the counter-clockwise direction. We then want to construct a loop that goes around this point once. We will let

$$\gamma_1(t) = \begin{cases} c(t) = \left(-2+\frac{1}{4}i\right) + \left(t-\frac{1}{3}\right)t, & 0 \le t \le \frac{1}{3}; \\ c(t) = -2 + \frac{1}{4}e^{6\pi i\left(t-\frac{1}{3}\right)}, & \frac{1}{3} \le t \le \frac{2}{3}; \\ c(t) = -2 + \frac{1}{4}i + \left(t-\frac{2}{3}\right), & \frac{2}{3} \le t \le 1. \end{cases}$$

Our base point is travelling to the point $c = -2+\frac{i}{4}$, around a circle centered at (-2,0) with radius $\frac{1}{4}$, and then back to the origin. We now want to observe the permutation of our period four points in Figure 18.

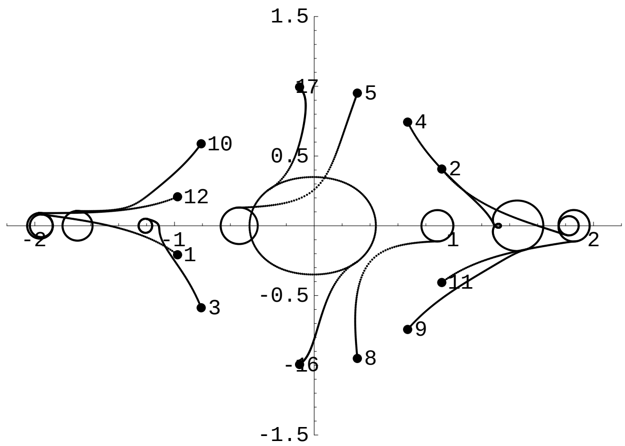

FIGURE 18. z-plane

If we follow the path of the period four points from $c=0$, we notice that

$$\varphi(\langle\gamma_1\rangle) = (1,12)(2,11)(4,9)(6,7)$$

12.3. **Permutations Induced by $\langle\gamma_2\rangle$.** We have that $\langle\gamma_2\rangle$ is the homotopy class of loops contains the ramification point $c_2^* = -\frac{5}{4}$. We then want to construct a loop that contains this point. We will let

$$\gamma_1(t) = \begin{cases} c(t) = -1 + \left(t - \frac{1}{3}\right) t, & 0 \le t \le \frac{1}{3}; \\ c(t) = -1.25 + .25 e^{6\pi i \left(t - \frac{1}{3}\right)}, & \frac{1}{3} \le t \le \frac{2}{3}; \\ c(t) = -1 + \left(t - \frac{2}{3}\right), & \frac{2}{3} \le t \le 1. \end{cases}$$

Our base point is travelling to the point $(-1, 0)$, around a circle centered at (-1.25,0) with radius $\frac{1}{4}$, and then back to the origin. We now want to observe the permutation of our period three points in Figure 19.

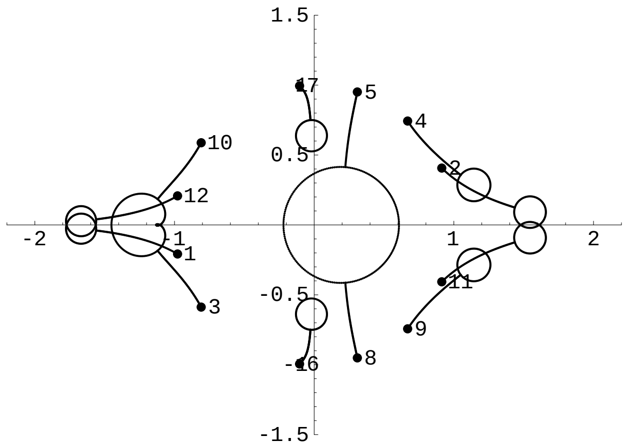

FIGURE 19. z-plane

We can now follow the path of our period three points. We see that

$$\varphi\left(\langle\gamma_2\rangle\right) = (1, 12)(3, 10)(5, 8)$$

12.4. **Permutations Induced by $\langle\gamma_3\rangle, \langle\gamma_4\rangle, \langle\gamma_5\rangle, \langle\gamma_6\rangle$.** For the remainder of the permutations, we will just state the permutations and display the corresponding mappings in the z-plane.

$$\begin{aligned} \varphi\left(\langle\gamma_3\rangle\right) &= (1, 10)(2, 8)(3, 4)(5, 7) \\ \varphi\left(\langle\gamma_4\rangle\right) &= (3, 12)(5, 11)(6, 8)(9, 10) \\ \varphi\left(\langle\gamma_5\rangle\right) &= (1, 2, 4, 7) \\ \varphi\left(\langle\gamma_6\rangle\right) &= (9, 11, 12, 6) \end{aligned}$$

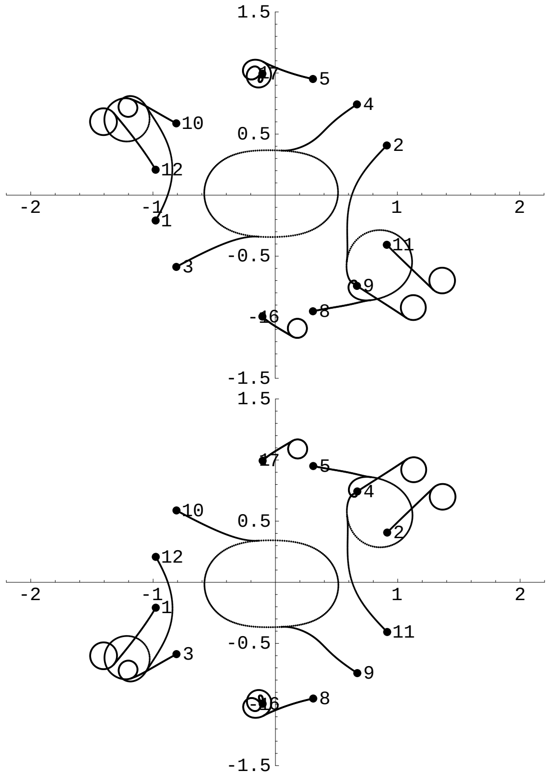

FIGURE 20. $\varphi(\langle\gamma_3\rangle), \varphi(\langle\gamma_4\rangle)$

Part 5. Further Research

The original intent of this research was to study the permutations of the period five points of the quadratic polynomial. As we saw throughout this research paper, our permutations became increasingly complicated. After an academic year of working on my research, I realize that I have only begun to skim the surface. While I have had a chance to study the permutations of the fixed, and period two points of the family of quadratic polynomials, I have only just started to explore the permutations of the period three and four points.

In dealing with the permutations of period three and four points, there are still many questions worth exploring. I want to continuing looking at the permutations from a group theory vantage point. I am still unclear as to what type of subgroup my permutations creates, and I want to further investigate the subgroup and its elements. I also want to continue looking at the products of the generators of the fundamental group. I have begun to notice something peculiar happening with the

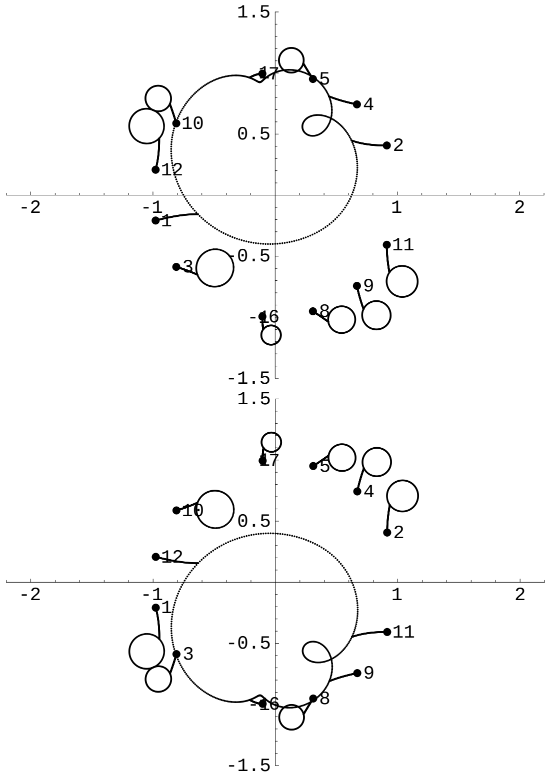

FIGURE 21. $\varphi\left(\langle\gamma_5\rangle\right), \varphi\left(\langle\gamma_6\rangle\right)$

permutations of the period four points of the quadratic, but have yet to have time to fully explore the ideas.

Lastly, I have yet to determine a manner in which to tie all my research into generalizations. There are connections that I need to develop in order to link the permutations of various period n points. I think I will have a better understanding of the workings of the permutations of the periodic points of the quadratics once I further investigate the permutations of periodic points of a higher iteration.

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